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Simulation of convection currents set up by jets and plumes in fluids of varying stratification

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ABSTRACT. Tollmien's analysis for velocity and cone angle of jets has been briefly examined. The jet dispersion was observed in a glass tank filled with water and/or brine. The angle of the cone of dispersion was observed to be the same whatever was the variation in stratification and density.

For observation of plumes, experiments were also carried out in the extended glass tank filled first with water and then with salt solution stratified in density layers. Isolated volumes of concentrated salt solution, suitably coloured with dye, were released from rest from above, and the dispersion was observed and photographically recorded.

The main discovery concerning a stratified fluid was that the cone of the plume (before it dispersed horizontally) tended to make a narrower 'neck', as in contrast with plume dispersion in an unstratified fluid.

As in the case of an unstratified fluid, the dimensional analysis equations (*see* Scorer, Richards, etc) were approximately obeyed initially, except that the values of the constant parameters (*e.g.*, n, C, etc.) were found to be different. A short note is included in this paper on bubble dynamics as was studied originally by Taylor.

1. Notation

The important notation used in the text is listed here below. Other notation used will be defined where appropriate.

Jet

U_1 = initial velocity
U_z = velocity at distance z
c = dimensionless constant
d = orifice diameter
z = distance from orifice
r = radius
$U_{mean} = k\, U_z$
k = 0·2
z, x = jet axes
α = semi-angle of jet
θ = angle of inclination of jet

Plume

W = rate of rise of plume front
C = universal dimensionless constant
g = gravitational acceleration
$\overline{B}$ = mean buoyancy $= \triangle\rho/\rho$
ρ = density
r = radius of plume
z = height of plume front
n = z/r
V = volume of plume $= mr^3$
m = $\frac{3}{4}\pi$ (for a sphere)
R = $g\, V_1\, \overline{B_1}$ (where suffix 1 refers to initial condition)
k = $\dfrac{m^{\frac{1}{2}}}{2nC\, R^{\frac{1}{2}}}$

2. Introduction

From the principle of Archimedes, one learnt that when a body is immersed in a fluid, it is subjected to an upward force. It is only in comparatively recent years, however, that experiments have been carried out to study the behaviour of a fluid released inside another fluid.

Much research effort has been made recently to understand this problem of convective mixing. In meteorological problems, the dispersion of smoke (for example) has come under study. Rational and empirical researches on dispersion of smoke jets and plumes in the atmosphere have led scientists to study this type of behaviour by analogy in incompressible fluids. (*see* the work of Richards 1960, 1961 and earlier work of Scorer 1958, Taylor 1934 etc.)

The dispersal characteristics of a jet for example may be investigated under different conditions. This may have a bearing on a flame jet in which the combustible is discharged at high speed.

The idea of dispersion may be first characterised by considering water discharging into air. Unless the water droplets are very small, the air is not appreciably affected. Smoke being discharged under pressure into the air on the other hand causes turbulent mixing of the fluids, and this finally grows as a plume when only buoyancy forces become more dominant.
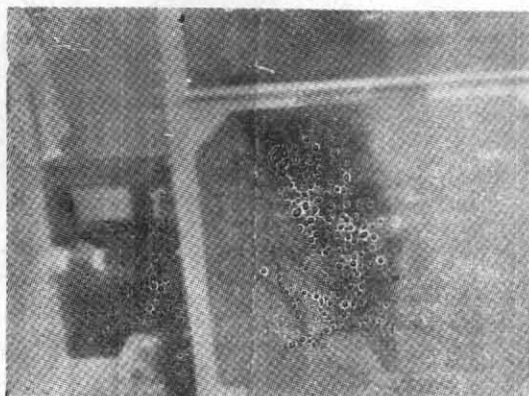
Plate 1

The photograph shows a jet dyed liquid dispersing into an extended fluid. For range of densities explored, the cone angle remained approximately the same for all cases $(a \simeq 15°)$

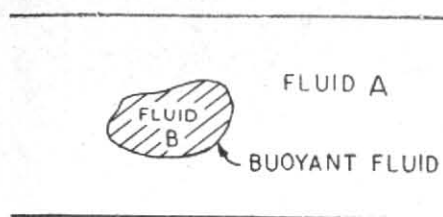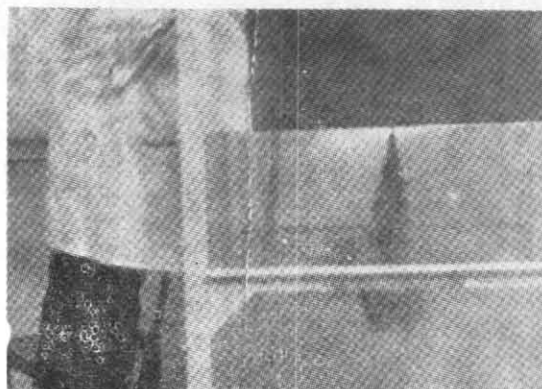Plates 2 & 3 . Photographs show a plume developing in a very strongly stratified fluid. Note the original 'narrow neck' before the dyed liquid reaches its own density layer and spreads out horizontally. Some of the fluid penetrates the 'mushroom' as well.
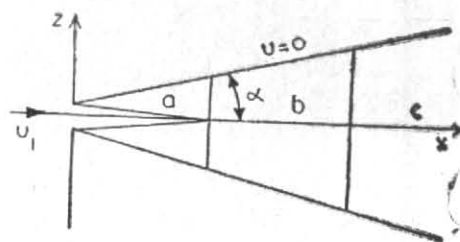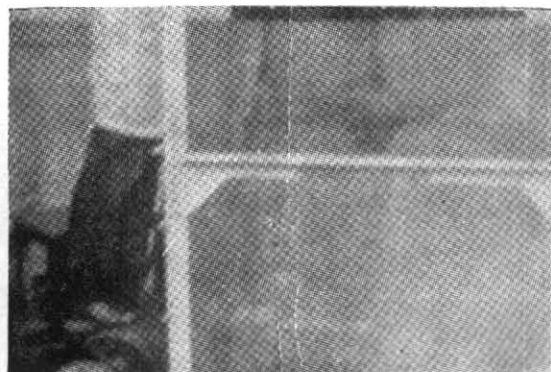

Fig.1 . A buoyant fluid immersed in another extended fluid

Fig. 2. Typical jet dispersion from a circular orifice. The semi-angle a is approx. 12°

Plate 2

Fig. 3. Kuethe's experimental profiles of a jet at various distances from the nozzle. The profiles follow a normal distribution.

A vertical jet in the initial stages was observed to have the following properties —

(1) Velocity decreased along the axis of projection.

(2) The velocity profile followed approximately, a Gaussian curve (normal distribution), being at its greatest at the axis, and diminishing rapidly towards the jet boundary.
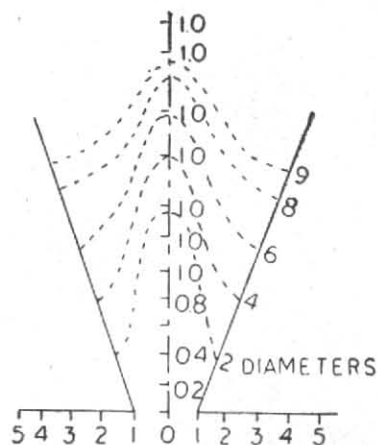
(3) The cone angle of the jet was in a highly turbulent condition, and absorbed the surrounding fluid increasingly from the orifice.

The third observation is of importance for the engineering problem of combustion.

On a different scale, plumes were also observed to have these characteristics.

3. Mixing of jets

3.1. *The mixing region and the effect of shear stress*

There are certain remarks that should be made concerning the mixing caused by a jet. When the jet left the orifice, the momentum (or kinetic energy) was slowly dispersed into the turbulent zone. A boundary separated the original jet from the surrounding fluid. This boundary was unstable, and broke into small vortices. The axial velocity decreased due to energy exchanges.

The effect of shearing stress has been usually ignored. But in 1925, Prandtl (1925) gave an expression:

$$\sigma_{zx} = \rho l^2 \left(\frac{du}{dz} \right)^2 \tag{1}$$

where,

σ_{zx} = shear stress
Z = direction of jet = $z(x)$
ρ = density
u = mean velocity
l = mixing length.

The mixing length l varied proportionately with the mean distance the fluid could move in the x-direction without losing momentum. In an infinite region with free turbulence, l may be assumed to be approximately constant for each section, but has to be determined by experiment.

3.2. *Tollmien's solution for symmetrical jet*

Tollmien (1926) gave the empirical relation for velocity along a jet as

$$U_z = \frac{cd \, U_1}{x + 1 \cdot 9 \, d} \tag{2}$$

where,

U_1 = initial velocity
c = dimensionless constant
d = orifice diameter
x = distance from orifice

His analysis gave a value for the cone angle as given by

$$r = 0 \cdot 214 \, x \tag{3}$$

such that the semi-angle α was approximately $12°$.

Kuethe (1935) carried out jet experiments, and found that the geometrical shape of velocity profiles remained similar for considerable distances along the jet. And he effectively verified Tollmien's original velocity relationship.

Plate 4 . Photograph shows a plume developing in more weakly stratified fluid. Again there is a 'narrow neck,' but not much fluid penetrates the giant 'mushroom'. The plume is a good inverted model of a mushroom cloud of a H-bomb explosion.
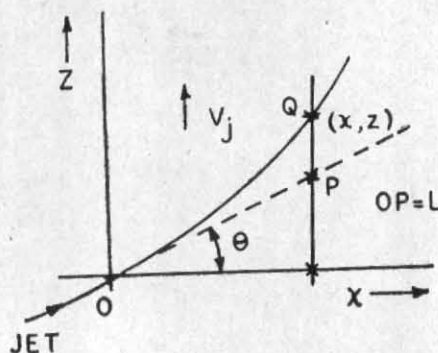

Fig. 4. Change of velocity direction for angled jet

In this paper, the effect of temperature variation is ignored, but in general this would also have to be considered.

3.3. *General formula for angled jet*

Consider a jet discharging into a fluid as analysed in Fig. 4.

For a jet, Tollmien found that the mean velocity U_{mean} was given by

$$\frac{U_{mean}}{U_z} = k \tag{4}$$

where,

k = constant $\simeq 0 \cdot 2$

If U_{mean} is the mean velocity at P (*see* Fig. 4) then

$$U_{mean} = \frac{ad \, U_1}{L + 1 \cdot 9d} \tag{5}$$

where,
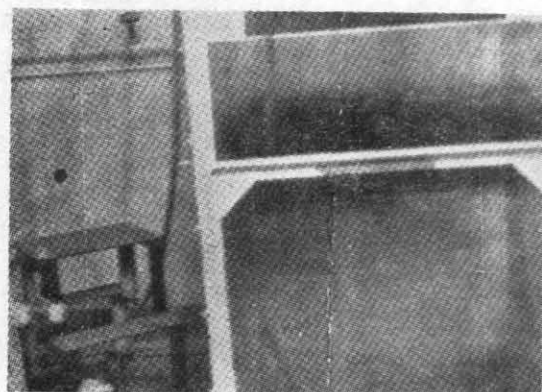
$OP = L$ and $a = ck =$ constant.

Plate 5 : Photograph shows the simulation of fog. The fluid
was stably stratified, and the dyed fluid allowed to find its
own density level, and settle overnight into the 'fog region'.
Once such a fog is formed, the conditions are often stable,
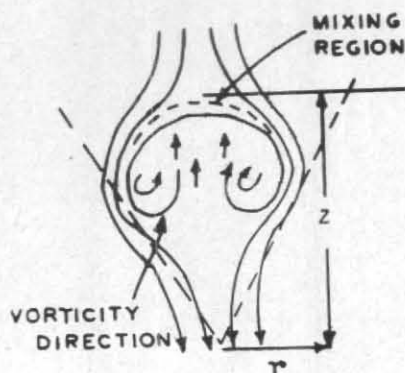and dispersion takes considerable time.

Fig. 5

A typical cross-section of a plume dispersing in an
unstratified fluid (Scorer 1958)

Writing

$$L = x/\cos\theta \qquad (6)$$

$$U_{mean} = \frac{a\,d\,U_1\,\cos\theta}{x+1\cdot9\,d\,\cos\theta} \qquad (7)$$

Horizontally,

$$\frac{dx}{dt} = U_{mean}\cos\theta \qquad (8)$$

Vertically,

$$\frac{dz}{dt} = V_j + U_{mean}\sin\theta \qquad (9)$$

where V_j is the original velocity of the extended
fluid.

$$\frac{dz}{dx} = \frac{V_j + U_{mean}\sin\theta}{U_{mean}\cos\theta} \qquad (10)$$

$$= \frac{V_j\,(x+1\cdot9\,d\,\cos\theta)}{a\,d\,V_1\cos^2\theta} + \tan\theta \qquad (11)$$

$$\text{or, } z = \frac{V_j\,x\,(x+3\cdot8\,d\,\cos\theta)}{2\,a\,d\,V_1\cos^2\theta} + x\,\tan\theta \qquad (12)$$

This analysis holds true for a pure jet discharging
into uniform surroundings.

When there are density variations, the buoyancy
forces have a larger part to play in the fluid motion.
The accelerating force is given by $\overline{B}\,g$,

$$\text{where } \overline{B} = \frac{\triangle\rho}{\rho} \qquad (13)$$

As stated earlier, the effect of this becomes impor-
tant when the jet is slowly transformed into a
plume.

This analysis has been presented for the
sake of completeness. Reference should be made to
standard work on this subject. The remainder
of the paper will deal with buoyant convection
caused by density variations.

4. Convective penetration of a fluid by buoyant plumes

4.1. Dispersion of plumes in an unstratified fluid (Scorer's Theory)

It is assumed that the viscosity is negligible
both for prototype and model. For air and water,
this is a reasonable assumption, and experiment
shows that Reynolds' number considerations may
be ignored at least as a first approximation.

The size of the region can be represented by
some measurement radius r of the largest horizon-
tal section drawn through the region.

For small density differences, if we consider
the mean buoyancy $g\,\overline{B}$ and the distance r, then
the vertical velocity W of the cap of the region is
compounded from these.

$\overline{B}$ is the density anomaly of the form

$$\frac{\rho - \rho^1}{\rho} = 1 - (\rho^1/\rho) \qquad (14)$$

and has no dimensions. $(\rho - \rho^1)$ is the change
in density, and this may also be written as $\triangle\rho$.

The only way we may have a relation for W is as
follows —

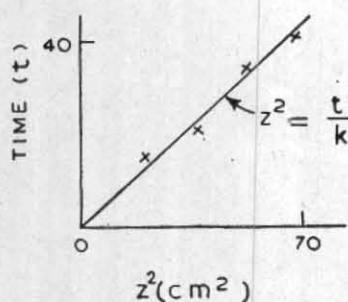$$W = C\,(g\,\overline{B}\,r)^{\frac{1}{2}} \qquad (15)$$
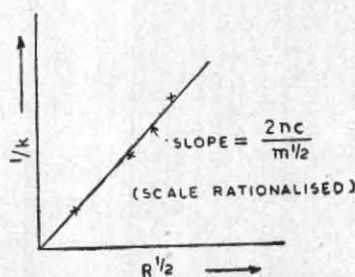
Fig. 6

Graph of t against Z^2

Fig. 7

Graph of $1/k$ against $R^{\frac{1}{2}}$

where C is a numerical constant. This has to be obtained by experiment as there is no other available theory.

From Fig. 5, we see that

$$z = nr \qquad (16)$$

where n is another constant determined by experiment. If the volume and buoyancy at a particular moment initially are V_1 and $g\,\overline{B}_1$, and

$$V = m\,r^3 \qquad (17)$$

Where m depends on the size and shape of the moving region, then since total buoyancy remains the same,

$$V\,\overline{B} = V_1\,\overline{B}_1 \qquad (18)$$

and hence

$$\overline{B} = \frac{r_1{}^3\overline{B}_1}{r^3} \qquad (19)$$

where

$$W = \frac{C\,(g\,r_1{}^3\,\overline{B}_1)^{\frac{1}{2}}}{Z} \qquad (20)$$

But W is given by the rate of change of Z, i. e.,

$$W = \frac{dz}{dt} \qquad (21)$$

Therefore,

$$Z^2 = 2nC\,(gr_1{}^3\,\overline{B}_1)^{\frac{1}{2}} \cdot t = \frac{2n\,CR^{\frac{1}{2}}}{m^{\frac{1}{2}}} \cdot t \quad (22)$$

where

$$R = gV_1\,\overline{B}_1 \qquad (23)$$

= buoyancy or weight deficiency (or excess).

For a given experiment, the quantities n, C, R and m are numerical constants (see Scorer 1958).

Hence $Z^2 = \dfrac{t}{k}$ (24)

where $k = \dfrac{m^{\frac{1}{2}}}{2n\,CR^{\frac{1}{2}}}$ (25)

We can perform an experiment and plot Z^2 against t, and from the graph verify the value of k (Fig. 6).

Evaluation of C

From the expression for k, we have

$$\frac{1}{KR^{\frac{1}{2}}} = \frac{2nC}{m^{\frac{1}{2}}} = \text{constant} \qquad (26)$$

If we plot $1/k$ against $R^{\frac{1}{2}}$ we shall obtain a series of points representing pairs of values k^{-1} and $R^{\frac{1}{2}}$. Each point would represent observations from one experiment.

Experiments can be performed to determine the total buoyancy (or excess) by placing a measured quantity of salt solution in a spherical cup.

The slope of the graph can be used to verify an experimental value of $2nC/m^{\frac{1}{2}}$.

In this expression, n is of the order of 4.0, and m is $4/3\,\pi$ (by assuming a spherical region). From experiments performed by Sen Gupta, C was found to be 1.10.

Richards has performed similar experiments, and his value for C was 1.26 (Richards 1962).

4.2. Dispersion in a stratified fluid

When the fluid was stratified in density layers, it was observed that dispersion took place in a different form (see Fig. 8). There was a narrowing of the 'neck' before a horizontal spread.

The narrowing of the cone angle may be attributed to internal waves forming in the stratified fluid while the plume is ascending or descending. As it penetrates the stratified layers, waves are set up as shown in Fig. 9. These waves tend to 'squash' the plume in the vertical plane. The region A—B shows this in the diagram.
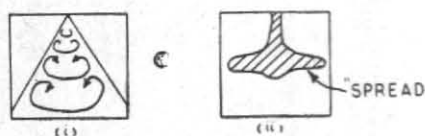
Fig. 8. Typical plume dispersion. (*i*) In an unstratified surrounding. Note the descent in a wide-angled cone. (*ii*) In a strongly stratified fluid. The cone angle *is* much smaller initially and then spreads out horizontally once the plume reaches a density at which it can penetrate no further.
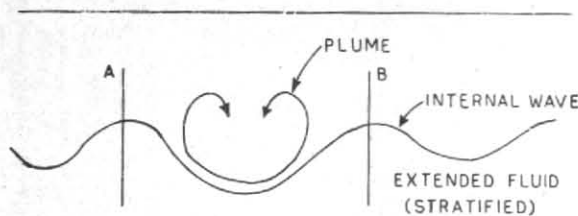

Fig. 9. Plume descending in a stratified fluid, and thereby setting up waves similar to those obtained when, for example, air flows over a mountain barrier.
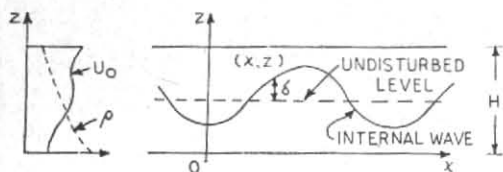

Fig. 10. Formation of internal wave in a stratified fluid

Since the plume mixes with surrounding fluid as it moves, its density is also altered. When the density is the same as the surrounding fluid, it spreads out in a horizontal plane.

The physical conditions can be altered to give any number of different flows. If the extended fluid is only very weakly stratified, then the flow pattern would behave very similarly to that of an unstratified fluid.

4.3. *Analogy of critical condition for wave formation*

The critical condition for wave formation can be further investigated for two-dimensional steady-state flows in a stratified fluid.

In this instance, the simplest wave equation is of the form :

$$\nabla^2 \delta + G_o \, \delta = 0 \qquad (27)$$

Here,

(x, z) = space co-ordinates

$$\nabla^2 = \frac{\partial^2}{\partial_x{}^2} + \frac{\partial^2}{\partial_z{}^2}$$

$$G_0 = \frac{g\beta_0}{U_o{}^2}$$

$\delta =$ wave amplitude

$$\beta_0 = -\frac{1}{\rho}\frac{\partial\rho}{\partial z}$$

$\rho =$ density, $H =$ depth of channel, and

$g =$ gravity, $U =$ undisturbed velocity.

If for theortical purposes, G_o, is made constant, then it may be shown that for waves to form in a horizontal stream, the required condition is :

$$G_0 > \frac{\pi^2}{H^2} \qquad (28)$$

This shows that for stratifications weaker than $G_0 = (\pi^2/H^2)$ one would not obtain a wave solution. A similar critical condition would also exist for the general three-dimensional plume moving in a stratified fluid.

4.4. *Experiments*

For various conditions of density and stratification, plumes were observed by releasing dyed heavy liquid (salt solution) from rest from the surface. By knowing initial densities, the mass excess could be calculated.

Records were taken at regular intervals by photography, and values of Z, r and t were noted.

Stratification was achieved in the tank by introducing layers of brine from underneath, each successive layer being denser than the preceding one (*see* Fig. 11).

4.5. *Bubble convection*

The concept of buoyant convection can be introduced through the formation of bubbles (Davies 1950, Taylor 1934). If a statistically unstable layer occurs in a fluid, such that it is not horizontally uniform, then the disturbance sets up oscillatory motions.

We can illustrate this phenomenon in simple form. The motion involved will not be dissimilar to cellular convection, especially if there are temperature considerations.

If a hot bubble is released, the hottest part will rise to the top of the bubble, and there will be a sharp discontinuity of temperature at the bubble cap. The horizontal density gradients will cause a generation of vorticity leaving behind a turbulent wake (Fig. 12).

In any system of bubble dynamics, we assume that bubbles of any size may form.

We assume that erosion and 'form drag' tend to make bubbles ascend without appreciable acceleration. Then as was shown by Scorer, the velocity of ascent, W, is determined only by size of bubble and buoyancy $\overline{B}$ (defined as the ratio of difference in density between the bubble and surroundings to the density of surroundings themselves). We may write the size as a linear dimension .

Then, $W \propto (g \overline{B} r)^{\frac{1}{2}}$ (29)

This is a similar formula to the one used for smoke plumes, except that in this case $\overline{B}$ is constant, and the constant of proportionality is different.

Taylor (1934) and Davies (1950) carried out a simple calculation based on potential theory. This is the elegant branch of hydrodynamics which deals with pressure gradient and inertia forces, as they always balance.

Taylor and Davies (1950) noted that bubbles rising through water have usually spherical caps and rugged lower surfaces. In hydrodynamics, we can often apply the classical theories to the *front* surface.

Taylor's theory applied to a sphere when gravity was neglected gave an expression for the pressure at an angle δ from the forward stagnation point.

$$p_0 - \frac{9}{8} \rho \ W^2 \sin^2 \delta \qquad (30)$$

p_0 being the pressure at stagnation point and W the velocity of bubble in stationary fluid.

The pressure due to the weight of fluid is given by

$$p_0 + \rho g z, \qquad (31)$$

Z being the depth below the stagnation point. We have

$$Z = r (1 - \cos \delta) \qquad (32)$$

To balance, the inertia forces represented by $-(9/8) \rho W^2 \sin \delta$ must be equal to the effect produced by gravity, $\rho g z$

i.e., $$\frac{9}{8} W^2 \sin^2 \delta = gr (1 - \cos \delta) \qquad (33)$$

providing $W = \frac{2}{3} (gr)^{\frac{1}{2}}$ (34)

because $\sin^2 \delta = 2 (1 - \cos \delta)$ approximately for δ less than about 35°. If this formula is applied to oil bubbles of density ρ^1, we would have

$$W = \frac{2}{3} \ (g \frac{\rho - \rho^1}{\rho} r)^{\frac{1}{2}} \qquad (35)$$
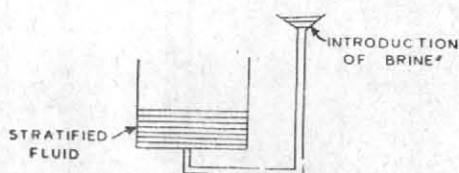

Fig. 11. Diagram shows how the fluid was stably stratified in density layers

Plate 6

Time exposure of a dense cylindrical white cloud descending through a water tank containing small white spherical neutrally buoyant droplets. The velocities of the surrounding water during the exposure are revealed by proportionate displacement of each droplet.

(The author is extremely grateful to Dr. J.M. Richards for permitting to publish this photograph taken during one of his very impressive experiments.)
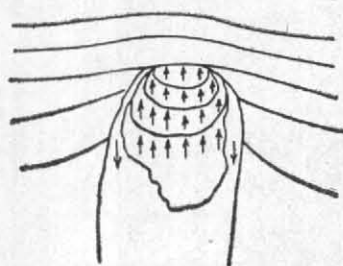

Fig. 12. Bubble with turbulent wake

Fig.13. Frontal profile of a rising bubble

Taylor also explained why the top of even large bubbles remained spherical. He did this by assuming that the pressure in the bubble was constant, and applied Bernoulli's equation for steady flow relative to the bubble (Scorer 1958).

5. Concluding Remarks

This research has been stimulated by the present day need for tackling the general problem of pollution.

The general motions of the atmosphere have not been fully understood owing to the large number of variables involved.

This paper attempts to obtain simplified equations in which the number of non-dimensional numbers are reduced to one. But there are also problems of diffusion of pollution and of momentum to eddying motion in the atmosphere, so that Reynlod's number corresponding to eddy transfer processes is involved.

The static stability of the atmosphere and the temperature excess of effluents introduce forces due to buoyancy, and this work has involved the use of Froude numbers.

The vertical profile both of wind and temperature are infinitely variable, and it would be remarked that they cannot truthfully be described by a small number of parameters.

Forces due to efflux velocity and horizontal wind velocity at the point of emission are also important in some cases. Such factors as ratio between height of orifice, size of orifice and distance from it are also relevant.

Probably the most important factor in removal of pollution is the horizontal wind. This has not been discussed in this paper.

Effect of stable stratification

On a sunny day, there is a strong convection in the atmosphere. As a result, the pollution is continuously stirred in a region several hundred feet deep.

Thus the effluents from tall chimneys can often be found to have negligible effect on the ground, whereas pollution from traffic and houses accumulates and is trapped in the lowest layer.

As the first part of this work shows it is desirable to obtain accurately the non-dimensional parameters involved (e.g., C, n, m, etc). An accurate estimate of these coefficients would give a fairly substantial basis for experiment, involving both model and prototype. The simple theory breaks down however, for general cases involving stable stratification.

Acknowledgements

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