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Fourier expansion techniques in objective analysis

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ABSTRACT. Cressman's objective analysis procedure has been interpreted in the light of generalised Fourier transforms and is shown as a practical method of local point Fourier analysis. A procedure to construct amplitudes of the field of analysis from the grid point pseudo observations and actual station locations is than discussed. The objective analysis field derived from these amplitudes compares favourably with the subjective analysis. Possible modifications are suggested for an attempt to decrease the strong bias of the pseudo observations over the actual station data in the analysis.

1. Introduction

Objective analysis methods have gained importance in Meteorology after the advent of numerical weather prediction in which we solve a set of dynamic equations for tendency with respect to time for a number of field parameters. The most widely used method for objective analysis is by Cressman (1959), though there are other methods in operational use (Gandin 1963, Kruger 1967). In this study we shall be concerned with Cressman's technique.

2. Cressman's analysis procedure and generalised Fourier transforms

2.1. From the weighted average correction, first reported by Bergthorssen and Döös (1955), Cressman evolved an iterative scanning procedure whereby the guess field on a regular grid point was modified by reported observations in successive scans. The first guess value at each grid point was adjusted by all observations lying within a radius of influence D from the grid point. This adjusted value becomes the initial guess for a succeeding scan of smaller radius. The number of scans and radii are chosen to incorporate the finer details of the analysis. The adjustment is done by giving suitable weights to the departure of the current observation from the interpolated value of the guess field. Cressman used a weighting function $(D^2 - R^2)/(D^2 + R^2)$ where, D is the radius of the scan and R is the distance of the observation location from the grid point.

2.2. The pertinent questions which arise in this context are :

   (*a*) What is the role of the first guess field, and

   (*b*) How does it influence the final results ?

These problems were discussed by Petersen (1967), who found theoretical support for using the latest prognosis as the first guess. We note that the prognosis at the time of analysis has already been incorporated in the usable information of the previous observations. Consequently, it is an useful parameter for the first guess. Petersen (*loc. cit*) also demonstrated that for pure waves as input, with a relatively sparse periodic array of hypothetical observation points, the wave number transfer function of Cressman's procedure is appreciably broad. For a typical distribution of upper air reporting stations this procedure transfers about 40% of the input amplitude into other wave number regions.

2.3. Cressman's procedure may be intuitively considered in the light of generalised Fourier transforms. The following considerations are relevant.

Let the difference between the actual observation and interpolated first guess value at the observation location, represented by $S(x, y)$ be taken as a correction factor in a finite domain (L, B) around the grid point. We represent L as the length in the X-direction and B as the breadth in the Y-direction.

For the existance of a Fourier transform of $S(x, y)$ a sufficient condition is the convergence of the integral,

$$\int\limits_{-\infty}^{+\infty}\!\!\int |S(x, y)| \, dx \, dy \tag{1}$$

The idea motivating generalised Fourier analysis is to make this convergence possible. To this end we create a set of scaling functions $G_R(x, y)$,

**Fig. 1. Actual wind field on 1 July 1965
(00Z) at 500 mb**

where, R is a real number with the following properties :

(i) $$G_R (x, y) \to 0 \qquad (2)$$
$$R \to \pm \infty$$

As $R \to \pm \infty$, G_R approaches zero, and the product $S(x, y) G_R (x, y)$ becomes convergent and integrable. It will thus have a Fourier transform.

(ii) The second property of the scaling functions is that,

$$\text{when } R \to 0, \ G_R (x, y) \to 1 \qquad (3)$$

The sequence $G_R (x, y)$ defines a generalised function such that,

$$\lim_{R \to 0} \int\int_{-\infty}^{+\infty} G_R (x, y) S (x, y) \, dx \, dy = \int\int_{-\infty}^{+\infty} S(x, y) \, dx \, dy \qquad (4)$$

In this respect it is different from another well known generalised function, namely, the delta function.

A function with these properties is,

$$G_R (x, y) = e^{-R^2/D^2}$$

where, $R^2 = (x^2 + y^2)$ and D is any real number.

Let us consider the scaling function in Cressman's method from this point of view. If $R (= \sqrt{x^2 + y^2})$ becomes large and approaches its maximum value D, the scaling function $G_R (x, y)$ approaches zero. Secondly, as the distance decreases and ultimately the station location coincides with

the grid point , $G_R (x, y)$ becomes one. This implies that the effect of station observations in modifying a grid point value falls off rapidly, and vanishes completely as the distance from the station to the grid point approaches the scan distance D. The scaling function of Cressman's procedure has the form exp. $\{-2R^2/(R^2 + D^2)\}$ whose expansion upto the first two terms yields the customary weighting function $(D^2 - R^2)/(D^2 + R^2)$.

Let us consider a circular domain of scan radius D around a grid point. The fundamental wave-numbers in X and Y directions are $k = 2\pi/D$; $l = 2\pi/D$, and the correction factors in station locations are $S(R, \theta)$, where, (R, θ) represent the polar coordinates of a station. The Fourier transform $H(m, n) = a(m, n) + ib (m, n)$ in the domain is,

$$H(m, n) = \int_0^D \int_0^{2\pi} S (R, \theta) \frac{D^2 - R^2}{D^2 + R^2} \times$$
$$\times \exp. [-iR (m \cos \theta + n \sin \theta)] \, R \, dR \, d\theta \qquad (5)$$

where, m and n are the indices of the mth and nth harmonics of the fundamental wave numbers k and l respectively.

$$k = l = 2\pi/D ; \ R \cos \theta = X = kx ;$$
$$R \sin \theta = Y = ly$$

If $m = n = 0$

$$H (0, 0) = \int_0^D \int_0^{2\pi} S (R, \theta) \frac{D^2 - R^2}{D^2 + R^2} R \, dR \, d\theta \qquad (6)$$

An average of the weighted correction factors from station location is,

$$\frac{\int_0^D \int_0^{2\pi} S(R, \theta) \frac{D^2 - R^2}{D^2 + R^2} R \, dR \, d\theta}{\int_0^D \int_0^{2\pi} R \, dR \, d\theta} = \frac{H(0,0)}{\pi D^2} \qquad (7)$$

In Cressman's analysis the average weighted correction factors around a grid point is used to modify the grid point value. Consequently, we infer that the modifications are only with respect to $H(0,0)$, and the higher harmonics are not taken into account. To compensate for this in a practical way, the scan length was decreased in a sequential manner and the process was repeated for a fixed number of scans. The reporting stations in the circular scan area around a grid point must be uniformly scattered on all sides to get the best results. Moreover, if there are no common stations in the scan circles around two contiguous grid points, this method will not have continuity throughout the field.
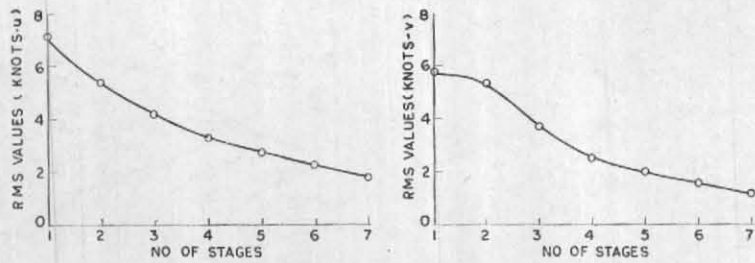
Fig. 2. R.M.S. error of computed field stages (a) U (b) V

TABLE 1

Stage	Number of Fourier components added	Description	Number of terms for least square analysis
I	1	$m=0$ $n=0$	1
II	2	$m=1\ 0$ $n=0\ 1$	3
III	4	$m=2\ 1\ 1\ 0$ $n=0\ 1\text{-}1\ 2$	7
IV	6	$m=3\ 2\ 2\ 1\ 1\ 0$ $n=0\ 1\text{-}1\ 2\text{-}2\ 3$	13
V	8	$m=4\ 3\ 3\ 2\ 2\ 1\ 1\ 0$ $n=0\ 1\text{-}1\ 2\text{-}2\ 3\text{-}3\ 4$	21
VI	10	$m=5\ 4\ 4\ 3\ 3\ 2\ 2\ 1\ 1\ 0$ $n=0\ 1\text{-}1\ 2\text{-}2\ 3\text{-}3\ 4\text{-}4\ 5$	31
VII	12	$m=6\ 5\ 5\ 4\ 4\ 3\ 3\ 2\ 2\ 1\ 1\ 0$ $n=0\ 1\text{-}1\ 2\text{-}2\ 3\text{-}3\ 4\text{-}4\ 5\text{-}5\ 6$	43

3. Method of Fourier analysis in this study

3.1. Motion in the atmosphere occurs on several scales, ranging from the smallest turbulent eddies to very large planetary waves. The instantaneous field is an en semble of infinite Fourier components. In meteorological processes our main aim is to represent the scale of Rossby waves. Thus we adopt a discrete Fourier analysis to depict the synoptic scales of motion to the maximum extent possible. This study consists of two parts :

(a) Obtaining the amplitude of the scalar guess-field by finite Fourier analysis, and

(b) Meshing the station observations with the first guess-field as in (a) with proper scaling functions. The procedures have been outlined in appendices I and II.

3.2. Case study 1

To find out how best the method detailed in Appendix I may be applied in the analysis procedure, we first computed the amplitudes of a subjectively analysed wind field at regular grid points. From these amplitudes, the grid point winds were reconstructed and the fit was examined by comparing the kinetic energy (K.E.) of the reconstructed field, with the K.E. of the initial field. These calculations were made on several 500 mb winds during the south-west monsoon. One case is discussed here.

The analysed wind data over a grid from 40° N to the equator and 65° E to 110° E at 2·5° intervals for 1 July 1965 at 500 mb (Fig. 1) was split into scalar U and V components. The fundamental wave lengths selected were 90 degrees along both X and Y (east-west and south-north) directions. The harmonics were added in seven stages as in Table 1 upto a number of 43. The R.M.S. error diminished at each stage with the addition of more terms in the analysis of U and V fields as shown in Fig. 2. The computed K.E. was 95% of the K.E. in the actual grid point data input. The actual and computed wind fields (Fig. 1 and Fig. 4 respectively) show good agreement.

3.3. To compute the amplitudes of the wave components from irregularly spaced observations,
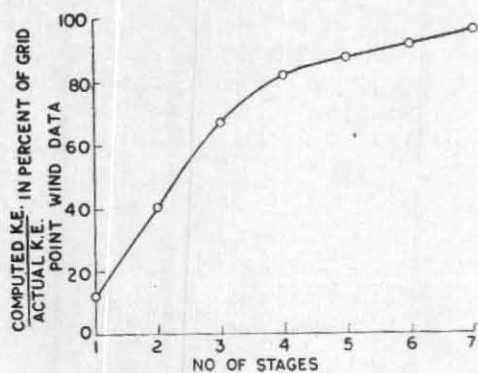
Fig. 3

Percentage of computed/actual kinetic energy
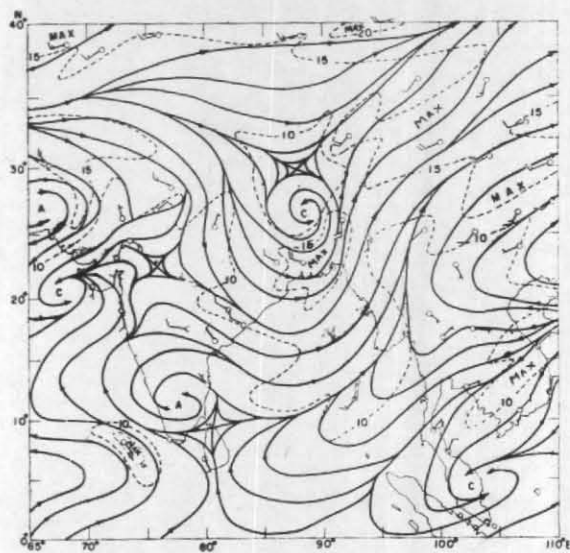field in stages

Fig. 5

Subjective analysis of station observations on
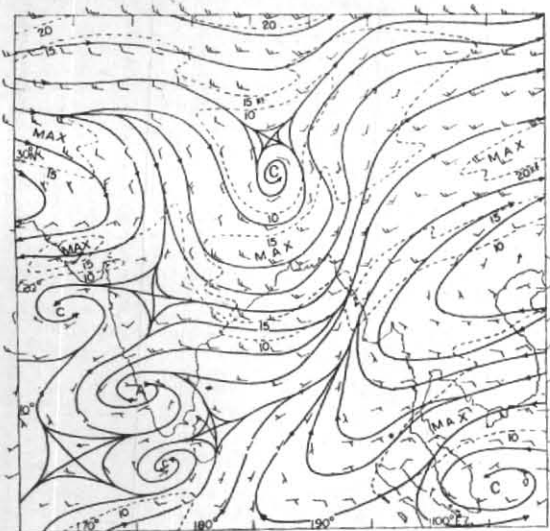1 July 1965 (12 Z) at 500 mb

Fig. 4

Computed wind field on 1 July 1965 (00 Z) at 500 mb

Fig. 6

Analysis by Fourier method of wind field on 1 July
1965 (12 Z) at 500 mb

the first guess field at regular grid points may be considered as pseudo-observation points. They are in addition to the actual observations at reporting stations to provide a dense network. The computations are simplified by taking the fundamental wavelengths in the X and Y directions as the length and breadth of the domain. Details are shown in Appendix II.

3.4. Case study 2

The first guess grid point values of wind data on 1 July 1965 (00Z) at 500 mb and the station observations (Fig. 5) on 1 July 1965 (12 GMT) 500 mb were used to construct the amplitudes of the field as described in Appendix II. As the harmonics were added in stages (Table 1) the small scale features appeared progressively in stages. The total K.E. at station locations computed from the amplitudes of U and V of the reconstructed field was about 84% of the total K.E. contained in the initial data. The objectively analysed winds for the situation under study is shown in Fig. 5. There was general agreement between the analysis shown in Fig. 5

(subjective) and Fig. 6 (objective) from synoptic point of view.

There is little doubt that the pseudo-observations give a strong bias to the analysis. This is probably because we have considered 323 pseudo-points (17×19) compared to only 50 station observations. The weighting procedure for computing the area integral may need revision to give more weightage to station observations. In future experiments, the number of pseudo points will be reduced.

4. Conclusion

The study shows that Fourier expansions may be used in objective analysis. In this method the properties of the field as a whole are considered for computing the amplitudes once the grid point first guess-field is suitably scaled by the station observations.

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Appendix I

Method of representing a scalar field by a finite double Fourier series in a rectangular domain

For numerical computations the domain is a lattice of grid points at grid intervals h and $S(I, J)$ is the analysed scalar value at a grid point (I, J). The coordinates are $x_I = (I-1)h$ and $y_J = (J-1)h$ from the origin. Let $H(m,n)$ be the amplitude of a harmonic wave component of the scalar field; m and n represent the mth and nth harmonics of the fundamental wave numbers k and l along X and Y directions with their respective limits M and N. Let,

$$\sum_0^M \sum_{-N}^N H(m,n) \exp. (i\ \overline{mkx+nly}) \triangle m \triangle n = F(I,J) \tag{1}$$

Theoretically the summation should be from $-M$ to $+M$ for m and $-N$ to $+N$ for n. But in practice to save computation time only limited number of harmonics are taken in both the directions. The final analysis of the field results from a best fit representation of all the included harmonics. Trial runs were made with limits from 0 to M for m and 0 to N for n but the orientation of the ridges and troughs in the initial field were not represented properly in the computed field. More harmonics were added by taking the limits for n from $-N$ to N. This brought the trough ridge orientation in the computed field to the correct phase.

Since the grid point values $S(I,J)$ are real we treat $F(I,J)$ as real. We have,

$$\sum_0^M \sum_{-N}^N [a(m,n) \cos\theta - b(m,n) \sin\theta] \triangle m \triangle n = F(I,J) \tag{2}$$

where, $H(m,n) = a(m,n) + ib(m,n)$ and
$$\theta = mkx_I + nly_J$$

$H(m,n)$ values are calculated by the method of least squares. For this the total error is E, where,

$$E = \sum_{I=1}^{I_{max}} \sum_{J=1}^{J_{max}} [S(I,J) - F(I,J)]^2 \tag{3}$$

is a minimum. I_{max}, J_{max} refer to the number of grid points along the X and Y directions respectively.

The necessary conditions for E to be a minimum are,

$$\frac{\partial E}{\partial a(m,n)} = \frac{\partial E}{\partial b(m,n)} = 0$$

This is put in the matrix form,

$$[T][Q] = [R] \tag{4}$$

where, $[T]$ is a square matrix, and $[Q]$ and $[R]$ are column matrices.

Construction of matrix $[T]$

The first row contains the elements $\cos(mkx_I + nly_J)$ and $-\sin(mkx_I + nly_J)$. For each set of values of m and n the summations for all grid points are obtained. Thus we have,

$$\sum_{I=1}^{I_{max}} \sum_{J=1}^{J_{max}} 1 \quad \sum\sum \cos kx_I \quad \sum\sum \cos(kx_I + nly_J) \quad ..\, 0 \quad \sum\sum - \sin kx_I ...$$

$$\begin{array}{ccccc} m=0 & m=1 & m=1 & m=0 & m=1 \\ n=0 & n=0 & n=1 & n=0 & n=0 \end{array}$$

The subsequent rows of the matrix are formed by multiplying the first row of elements by the second, third, etc elements. Then the summation at all grid points are obtained. Thus we have,

2nd Row $\quad \sum_{I=1}^{I_{max}} \sum_{J=1}^{J_{max}} 1. \cos kx_I \quad \sum\sum \cos kx_I \cos kx_I \quad \sum\sum \cos(kx_I + ly_J) \cos kx_I \quad ....$

3rd Row $\quad \sum_{I=1}^{I_{max}} \sum_{J=1}^{J_{max}} 1. \cos(kx_I + ly_J) \sum\sum [\cos(kx_I + ly_J)]^2 .... \quad ......$

and so on till the last row.

Column matrix $[Q]$

$[Q]$ is a column matrix of $a(m,n)$ and $b(m,n)$ as shown below :

$$[Q] = \begin{Bmatrix} a(0,0) \\ a(1,0) \\ a(0,1) \\ .. \\ .. \\ b(0,0) \\ b(1,0) \\ b(0,1) \\ .. \\ .. \end{Bmatrix} \quad \begin{array}{l} \text{for all } m, n \\ 0 \leqslant m \leqslant M \\ -N \leqslant n \leqslant N \end{array}$$

Column matrix $[R]$

This is another column matrix obtained by multiplying $S(I,J)$ by $\cos(mkx_I + nly_J)$ and $-\sin(mkx_I + nly_J)$ and summing up at all grid points for each pair of values (m,n) as below :

$$[R] = \begin{Bmatrix} \sum_{I=1}^{I_{max}} \sum_{J=1}^{J_{max}} S(I,J) \\ \sum\sum S(I,J) \cos kx_i \\ .. \\ .. \\ 0 \\ \sum\sum - S(I,J) \sin kx_I \\ .. \\ .. \end{Bmatrix} \quad \begin{array}{l} m=0; n=0 \\ m=1; n=0 \\ \\ .. \\ .. \\ m= \ ; n=0 \\ \text{for all } m, n. \\ 0 \geqslant m \geqslant M \\ -N > n \geqslant N \end{array}$$

The matrix Eq. (4) is then solved to obtain the solution vector Q and hence we evaluate the amplitudes $a(m,n)$ and $b(m,n)$. Knowing the coefficients a and b we get the grid point values $F(I,J)$ from (1).

Appendix II

The computations in Appendix I can be simplified if the fundamental wave numbers k and l are taken as $k = 2\pi/L$ and $l = 2\pi/B$, where L is the length and B is the breadth of the domain. Due to the orthogonality of trigonometric functions, the matrix T in Eq. (4) becomes a column vector of elements,

$$\left[\begin{array}{c} \sum_{I=1}^{I_{max}} \sum_{J=1}^{J_{max}} \cos^2 (mkx_I + nly_J) \\[2em] \sum_{I=1}^{I_{max}} \sum_{J=1}^{J_{max}} \sin^2 (mkx_I + nly_J) \end{array} \right] \quad \begin{array}{l} \text{For all } m, n \\ 0 \leqslant m \leqslant M \\ -N \leqslant n \leqslant N \end{array} \tag{1}$$

Hence,

$$a(m, n) = \frac{\sum_{I=1}^{I_{max}} \sum_{J=1}^{J_{max}} S(I, J) \cos (mkx_I + nly_J)}{\sum_{I=1}^{I_{max}} \sum_{J=1}^{J_{max}} \cos^2 (mkx_I + nly_J)} \tag{2}$$

and

$$b(m, n) = \frac{-\sum_{I=1}^{I_{max}} \sum_{J=1}^{J_{max}} S(I, J) \sin (mkx_I + nly_J)}{\sum_{I=1}^{I_{max}} \sum_{J=1}^{J_{max}} \sin^2 (mkx_I + nly_J)} \tag{3}$$