Orthogonal fields of temperature variation over Peninsular India

P. JAGANNATHAN and P. RAKHECHA

Institute of Tropical Meteorology, Poona

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ABSTRACT. Orthogonal spatial patterns of mean monthly air temperature distribution over Peninsular India are ABSTRACT. Orthogonal spatial patterns of mean monthly air temperature distribution over Peninsular India are
evolved and the seasonal variation and the inter-relationships between monthly patterns are presented. The chara changes in the intensity of this pattern were studied by power spectrum analysis. Significantly, the first C.P. shows a quasi-biennial wave.

1. Introduction

The variability of a quantity (F_{st}) in space and time can be described by resolving it into orthogonal components.

$$
F_{st}=a_t\ A_s+b_t\ B_s+\ \ldots \ldots \ldots \quad (1)
$$

where, the A_s , B_s etc denote space functions and their coefficients a_t , b_t etc are functions of time. As the space functions utilised are mutually orthogonal, *i.e.*, $\int A.B. ds = 0$, it will be possible to study the time-wise variations of the specified space field in terms of the component fields. Further, these functions could be used as independent factors in problems of statistical weather predictions. Similarly, if the time functions utilised are mutually orthogonal, the space variation of the field associated with the component time functions could also be studied.

For the study of problem described above Lorenz (1956) and subsequently a number of others developed the idea of using 'Empirical Orthogonal Functions' (E.O.F.), which are not of any pre-
determined form but are developed as unique functions from the data matrix. Grimmer (1963) utilised this technique to derive characteristic patterns out of temperature anomaly fields over Europe and for identifying the genetic character of the component fields. In the present paper, the characteristic patterns of temperature variation over the Peninsular India and neighbourhood, their seasonal variation and the inter-relationships between the successive months are presented.

2. Method of analysis

If T_{ij} is the mean air temperature of a specified month at the *i*-th station in the *j*-th year (25 stations, 50 years), the sequence of space fields representing distribution of temperature over the area can be written as ma'rix :

$$
T_{mn} = |T_{ij}| \tag{2}
$$

We try to represent these space field of temperature as a linear combination of a set of patterns, so that the matrix T_{mn} composed of the m time-series is transformed into a fresh set of m time-series such that the correlation between any two pairs of the latter series is 0. T_{mn} is operated on the transformation matrix Q_{nn} such that -

$$
T_{mn} \cdot (Q_{nn}) = P_{mn} \tag{3}
$$

where, $\sum_{i=1}^{n} P_{ri} P_{si} = 0 \quad (r \neq s)$

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It has been shown (Jagannathan' 1969) that the above transformation matrix also diagonalises the covariance matrix of temperature.

$$
\mathcal{C}_{mm} = \frac{1}{n} \left| \boldsymbol{T} \cdot \boldsymbol{T}' \right| \tag{4}
$$

where, the elements
$$
C_{ij} = \frac{1}{n} \sum_{r=1}^{n} T_{ir} T_{jr}
$$

$$
\text{and} \quad C_{ii} = \frac{1}{n} \sum_{r=1}^{n} \overset{2}{T}_{i_r}
$$

 T_{ir} represents the deviation from the mean, C_{ij} being the covariance between the temperature at the *i*-th and *j*-th stations and the diagonal elements C_{ii} are the variance of the temperatures at the *i*-th station.

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Station	Lat. $({}^{\circ}\mathrm{N})$	Long. $({}^{\circ}E)$	Elevatic (metres)		
Trivandrum	8°29'	76°57'	64		
Pamban	9°16'	79°18'	11		
Kodaikanal	$10^{\circ}14'$	77°28'	2343		
Mangalore	12°52'	$74^{\circ}51'$	22		
Bangalore	$12^{\circ}58'$	$77^{\circ}35'$	921		
Madras	13°00'	80°11'	16		
Marmugao	15°25'	73°47'	62		
Bellary	15°09'	$76^\circ 51'$	449		
Nellore	14°27'	79°59'	20		
Bijapur	16°49'	75°43'	594		
Hyderabad	17°27'	78°28'	545		
Masulípatnam	$16^{\circ}11'$	$81^{\circ}08'$	3		
Bombay	19°07'	$72^{\circ}51'$	15		
Aurangabad	19°53'	75°20'	581		
Chanda	19°58'	79°18'	193		
Gopalpur	$19^{\circ}16'$	84°53'	17		
Visakhapatnam	$17^{\circ}43'$	83°14'	3		
Indore	$22^{\circ}43'$	75°48'	567		
Pachmarhi	$22^{\circ}28'$	78°26'	1075		
Raipur	$21^{\circ}14'$	$81^\circ 39'$	298		
Cuttak	20°48'	85°56'	27		
Ahmadabad	23°04'	72°38'	55		
Veraval	$20^{\circ}54'$	70°22'	8		
Daltonganj	$24^{\circ}03'$	$84^{\circ}04'$	221		
Calcutta	22°39'	88°27'	6		

TABLE 1 $n \circ f$ station

Thus, C_{mm} . Q_{mm} Diagonal matrix (μ_1, μ_2, \ldots) \ldots , μ_m) Q_{mm} (5)

The diagonal elements $\mu_1, \mu_2, \ldots, \mu_m$ (eigen values) are derived by solving the "characteristic equation"

$$
\text{Det} \left[\mu U - C_{mm} \right] = 0 \tag{6}
$$

where, U is the unit matrix. Corresponding to each eigen value μ_j , the eigen vectors are obtained as the solution of the equation -

$$
C_{mm} \{ V_j \} = \mu_j \{ V_j \} \tag{7}
$$

All the eigen values and the eigen vectors are real and all the eigen values are positive. The eigen values are proportional to the mean-square accounted by each of the corresponding eigen vectors.

The solutions of the characteristic equation were obtained by a two-stage iteration process*. The first and highest eigen value and the corres ponding eigen vector are obtained as follows.

(i) Let
$$
e = \{ 1 \ 0 \ 0 \dots \}
$$
 be a unit column vector

$$
C_{mm} \quad e = \mu^{(1)} \{ V^{(1)} \}
$$

$$
C_{mm} \cdot \{ V^{(1)} \} = \mu^{(2)} \{ V^{(2)} \}
$$

...

$$
C_{mm} \cdot \{ V^{(l)} \} = \mu^{(l+1)} \{ V^{(l+1)} \} \qquad (8)
$$

This iteration is continued upto the stage-

$$
\text{Abs. } \{ \mu^{(l)} - \mu^{(l+1)} \} < \epsilon \tag{9}
$$

where, ϵ is an arbitrarily specified small value. Then $\{V^{(l)}\}$ is taken as the first eigen vector \mathbf{V}_1 with m elements and the corresponding $\mu^{(l)}$ is taken as the eigen value μ_1 . The space field associated with this vector is then computed as follows - Λ

$$
\begin{aligned}\n\bar{T}_{mn} &= \mathbf{V}_1 . \mathbf{V}'_1. \ \bar{T}_{mn} \\
\text{since } \mathbf{V}_1' &= \mathbf{V}_1^{-1}\n\end{aligned} \tag{10}
$$

 T_{mn} is subtracted from the original temperature field T_{mn} and the anomaly temperature field T_{mn} ⁽¹⁾ is obtained.

(*ii*) The anomaly field $T_{m n}$ ⁽¹⁾ is then treated in a similar manner and the second vector V_2 and the corresponding eigen value μ_2 are obtained in the same manner.

(iii) This process is repeated a number of times. A cut-off procedure is adopted to limit the computations at the stage, when the percentage of variance accounted by all the eigen vectors computed thus, is more than a specified percentage, say, 90 or 95 per cent.

3. Material for study

The data utilised for the study consists of 600 monthly fields of mean air temperature for each station for the period 1901 to 1950. The values were specified at a network of 25 stations distributed over the Peninsular India, and neighbourhood (see Table 1). Missing data (which did not exceed 1 per cent) were filled on the basis of collateral evidence from neighbouring climatological stations.

The characteristic patterns of the spatial variation of mean temperature for each of the twelve months were obtained as 'eigen vectors' from the temperature data matrix for the month and the total percentage variance accounted by the *i*-th pattern is 100 μ_i/ν where, ν is the total spacetime-variance.

Each characteristic pattern is a vector of 25 values corresponding to the 25 stations. Obviously, two patterns relating to the same month are orthogonal. Patterns given by these values for different months need not be orthogonal. The relation between the patterns for the different months will be examined by working out the CCs between successive months. It is seen subsequently

*The mathematical details and the computational procedure are being published elsewhere.

Orthogonal space fields of maan temperature ove Peninsular India

that some of the patterns relating to adjacent months are highly associated suggesting that the basic patterns are largely analogous, and implying a physical continuity.

4. Discussion of results

4.1. The percentage variance accounted by the first 6 characteristic patterns and the total number of such patterns required for accounting for 90 per cent of the variance are given in Table 2.

It is seen that the first six components account for 80-90 per cent of the variance in all the months except August and September, in which months they are able to account for only 75 per cent of the variance. The spatial patterns represented by the first three are shown in Figs. 1 to 4. The isolines are drawn at intervals of 5 units.

 (i) First Characteristic Pattern - The dominant pattern in all the months is the first eigen vactor, which accounts for 40 to 50 per cent of the spacetime variance during November to March, 59 per

cent in April and 70 per cent in June; the variance accounted is least about 30 per cent during July to October. The values of space fields I of Figs. 1 to 4 are high over the interior decreasing towards the coasts. The persistence of this feature is indicated by the high correlations for successive months given in Table 3.

The inter-monthly correlations are high, some of them being as high as 0.95 . The major features of the fields are somewhat similar as seen from the figures, even though there is a steady cycle of change in minor details of the characteristic pattern during the course of the year. This might suggest the presence of a common physical basis in this pattern throughout the year but requires more detailed study. The first vector could also be taken to represent the effect of continentality.

(ii) Second Characteristic Pattern - The second field is the best fit to the anomaly field after the first field has been eliminated, as such this field

TABLE 2

Percentage variance accounted by characteristic patterns

TABLE 3

Inter-monthly correlation for pattern I Jan Feb Mar $_{\rm Apr}$ May $_{\text{Jun}}$ Jul Aug $_{\rm Sep}$ $_{\rm Oct}$ $_{\rm Nov}$ $_{\mbox{\textbf{Dec}}}$ $_{\mathrm{Jan}}$ $+ 74$ $+0.97$ $+\cdot 91$ $+ \cdot 95$ $+\cdot 97$ $+\cdot 83$ $+\cdot 88$ $+ \cdot 80$ $+ .82 + .95$ $+ \cdot 94$ $+$ 81

TABLE 4

Inter-monthly correlation for pattern II

Inter-monthly correlation for pattern III

320

ORTHOGONAL FIELDS OF TEMPARATURE VARIATION

Orthogonal space fields of mean temperature over Peninsular India

adjusts the over and under estimates made by the first eigen vector. This pattern accounts for about 10 to 25 per cent of the space time variance, the lowest beng in the month of June with 7 per cent and the highest in February with 25 per cent. The inter-monthly correlations for this pattern given in Table 4 shows that there is fairly good similarity between patterns relating to the months November to February and also between July and August, while during the months April to July and during August to October, the monthly patterns oscillate between adjacent months, such that positive features are largely replaced by negative features vice versa in the subsequent month. and

(iii) Third Characteristic Pattern - The field. which is the best fit to the anomalies after the first two characteristic patterns have been eliminated accounts for 5 to 15 per cent of the variance, the lowest being in the month of June with 4

per cent and the highest being in the month of March with 16 per cent. The inter-monthly correlations given in Table 5 are generally poor.

(iv) Higher order Characteristic Patterns - The fourth characteristic pattern accounts for about 3 to 8 per cent; the fifth pattern accounts for about 2 to 7 per cent and the sixth pattern for about 2 to 5 per cent. The inter-monthly correlations for these patterns are given in Table 6.

Persistence - An attempt was next $4.2.$ made to study how the characteristic patterns of any month is related to the patterns of the previous month and as such multiple linear equations were worked out. Table 7 indicates the regression equations and multiple correlation coefficients. It is seen that the first C. P. of any month is highly correlated with first C. P. and in some cases by two C. Ps of previous month. The

 321

Regression equations	Multiple C.C.	Regression equations	Multiple C.C.
$V_{I11} = 1.2417V_{XII11} + 0.2091V_{XII11} - 0.0723$	- 86	$V_{\text{VII}:5} = 0.6762 V_{\text{VI}:5} + 0.4079 V_{\text{VI}:6} + 0.0124$	-80
$V_{I:s} = 0.8323 V_{XII:s} + 0.3483 V_{XII:s} - 0.0538$	-89	$V_{\text{VII}:6} = 0.5887 V_{\text{VI}:6} + 0.0059$.58
$V_{\text{I}:s} = -0.9500V_{\text{XII}:1} + 0.3456V_{\text{XII}:s} + 0.0591$.64	$V_{\text{VIII}:1} = 0.6565 V_{\text{VII}:1} + 0.0855$	-88
$V_{I14} = 0.7363 V_{XII14} - 0.0116$.74	$V_{VIII \; : \; 2} = 0.9550 V_{VII \; : \; 2} - 0.0290$	- 89
$V_{\text{I};\,5} = -0.3416V_{\text{XII};\,5} + 0.2847V_{\text{XII};\,6} + 0.0681$	-47	$V_{\text{VIII}:3} = 0.5445 V_{\text{VII}:3} + 0.3369 V_{\text{VII}:5}$	
$V_{I1:6} = -0.3573 V_{XII1:2} + 0.7000 V_{XII1:3} + 0.1676$	-78	$+0.5700VVII$: e+0.0025	.86
$V_{II;1} = 0.7074V_{I;1} - 0.4169V_{I;3} + 0.0052$.91	$V_{\text{VIII}:4} = 0.5615V_{\text{VII}:5} + 0.0002$	- 55
$V_{\text{II};2} = 0.4640V_{\text{I};1} + 0.7731V_{\text{I};2} - 0.1394$ $V_{\text{II};3} = -0.3551V_{\text{I};3} - 0.4390V_{\text{I};4} + 0.4699V_{\text{I};5}$.93	$V_{\rm VIII\;:\;s}\,=\,-0\cdotp5189\,V_{\rm VII\;:\;4} + 0\cdotp5039\,V_{\rm VII\;:\;6}$ -0.5016	.74
$-0.3793V_{1.6}-0.0041$ $V_{II;4} = 0.3841V_{I;3} + 0.5043V_{I;4} + 0.3716V_{I;5}$	-85	$V_{\text{VIII}:6} = 0.3900 V_{\text{VII}:3} + 0.4743 V_{\text{VII}:5}$ -0.0078	.60
$-0.2867V_1$: 6+0.0396	-80		
$V_{II;\,6} = 0.6002 V_{I;\,5} - 0.0139$	-57	$V_{\text{IX}+1} = 0.8160 V_{\text{VIII}+1} + 0.2549 V_{\text{VIII}+4}$ $+0.0164$	-93
$V_{\text{III i 1}} = 0.8526 V_{\text{II i 1}} + 0.0344$	-97	$V_{\text{IX}:2} = 0.9996 V_{\text{VIII}:1} + 0.7147 V_{\text{VIII}:1}$	
$V_{\text{III}:2} = 0.3877 V_{\text{II}:6} - 0.0370$.39	-0.1764	.87
$V_{\text{III}:3} = 0.9871 V_{\text{II}:2} + 0.0138$	-95	$V_{\text{IX}:3} = 0.3053 V_{\text{VIII}:3} - 0.7880 V_{\text{VIII}:4}$	
$V_{\text{III i,4}} = -0.7553V_{\text{II i,3}} + 0.5238V_{\text{II i,4}} + 0.0017$.90	-0.0338	-86
$V_{\text{III}:5} = 0.3900 V_{\text{II}:5} - 0.0296$.39	$V_{\text{IX}+4} = 0.6464 V_{\text{VIII}+3} + 0.4444 V_{\text{VIII}+4}$ $-0.4100 V$ VIII : $e^{-0.0298}$.88
$V_{\text{III};\,6} = -0.3582 V_{\text{II};\,4} - 0.3600 V_{\text{II};\,5}$ $-0.4800V_{II}$: $e^{-0.0014}$.70	$V_{\text{IX};1.5} = 0.5050 V_{\text{VIII};1.3} + 0.7200 V_{\text{VIII};6}$ -0.0114	.88
$V_{\text{IV}:1} = 0.8235V_{\text{III}:1} - 0.1871V_{\text{III}:2} - 0.0311$.97		
$V_{IV:2} = 0.7740V_{\text{III}:3} - 0.4330V_{\text{III}:4}$ $+0.3719V$ III: $5+0.0307$.91	$V_{\text{IX}:6} = 0.6053 V_{\text{VIII}:5} + 0.0036$.59
$V_{\text{IV}:3} = 0.4708V_{\text{III}:3} + 0.3283V_{\text{III}:4} + 0.0758$.54	$V_{\text{X}+1} = 0.9487 V_{\text{IX}+1} + 0.3010 V_{\text{IX}+2} - 0.0135$.93
$V_{\text{IV}+4} = -0.2973 V_{\text{III}+2} - 0.5482 V_{\text{III}+4}$ $-0.3148V$ III: $_6+0.0205$.72	$V_{X+2} = 0.7609 V_{IX+1} - 0.6902 V_{IX+2}$ $+0.4412V_{1X+3}-0.0722$.89
		$V_{\text{X}:3} = -0.3607 V_{\text{IX}:5} - 0.0501$	- 37
$V_{IV;5} = -0.3181V_{III;1} + 0.5117V_{III;4}$ -0.0443	.61	$V_{\text{X}+1} = -0.3576V_{\text{IX}+3} + 0.3844V_{\text{IX}+4}$.58
$V_{IV16} = 0.3636V_{\Pi11:2} + 0.0134$.36	$+0.0745$	
$V_{V+1} = 1.0041 V_{IV+1} - 0.0047$.95	$V_{\text{X}:5} = -0.3181 V_{\text{1X}:5} - 0.3300 V_{\text{1X}:5}$ $+0.3147$ $V_{\text{IX}:6}$ -0.0086	- 55
$V_{V12} = 0.6566 V_{IV12} + 0.3302 V_{IV14}$ $+0.5027V_{IV}:5+0.0593$.89	$V_{\rm X,16} = 0.4207 V_{\rm IX,13} + 0.0147$.41
$V_{V13} = 0.5800V_{IV12} + 0.5125V_{IV13} - 0.0171$.77		
$V_{\rm V\,\,:\,4} \,=\, -0\cdotp 5000\,V_{\rm IV\,\,:\,3} -0\cdotp 3144\,V_{\rm IV\,\,:\,4} + 0\cdotp 0533$	-59	$V_{\text{XI} \pm 1} = 0.7163 V_{\text{X} \pm 1} + 0.0633$.95
$V_{\nabla \pm 5} = 0.3796 V_{\nabla \pm 4} + 0.5002 V_{\nabla \pm 5} + 0.0523$.61	$V_{\text{XI}\;1:2} = 0.5820 V_{\text{X}\;1:2} + 0.4975 V_{\text{X}\;1:6} + 0.0005$	- 76
$V_{V16} = 0.5106 V_{IV14} - 0.0016$.49	$V_{\rm{XI}\;:\;s}\;=\;0\cdotp4600\,V_{\rm{X}\;:\;s}\,\text{---}0\cdotp5590\,V_{\rm{X}\;:\;4}\,\text{---}0\cdotp4100\,V_{\rm{X}\;:\;4}\,\text{---}0\cdotp241$	
$V_{VI;1} = 0.9613V_{V;1} + 0.0072$.97		.79
$V_{\text{VI}:2} = -0.3737V_{\text{V}:2} + 0.7425V_{\text{V}:2} - 0.0097$.84	$V_{\rm{XI}\;1.4}=0.6874V_{\rm{X}\;1.3}-0.5600V_{\rm{X}\;1.5}+0.0193$	87
$V_{\text{VI}:3} = -0.7555V_{\text{V}:2} - 0.3000V_{\text{V}:3}$ $+0.3591Vv:5-0.0022$.87	$V_{\text{XI}\pm 5} = 0.6700 V_{\text{X}\pm 2} + 0.5607 V_{\text{X}\pm 5} + 0.0032$.87
$V_{\text{VI}:4} = -0.4800V_{\text{V}:4} + 0.0370$.48	$V_{XII+1} = 0.9503 V_{XI+1} + 0.0085$.94
$V_{VI : s} = 0.4079V_{V : s} - 0.6600V_{V : s} + 0.0403$.78	$V_{\text{XII}:2} = 0.5122 V_{\text{XI}:2} - 0.6566 V_{\text{XI}:3} + 0.0462$.85
$V_{\rm{VII}\;:\;1}\,=\,1\cdotp0109\,V_{\rm{VI}\;:\;1}\!+\!0\cdotp2759\,V_{\rm{VI}\;:\;3}\!\!-\!0\cdotp0168$	- 93	$V_{\text{XII}:3} = 0.5527 V_{\text{XI}:2} + 0.5800 V_{\text{XI}:3}$ $-0.3800 V_{\text{XI}\,1.5} + 0.0264$.88
$V_{\rm\,VII\,\,:\,\,2}\,=\,0\cdotp5044\,V_{\rm\,VI\,\,:\,\,1}\cdots 0\cdotp6105\,V_{\rm\,VI\,\,:\,\,2}$ $-0.3147V_{\rm VI+3}-0.0204$.88		
$V_{VII \;:\; 3} = 0.6533 V_{VI \;:\; 4} - 0.0266$	-65	$V_{\text{XII}:4} = -0.4600 V_{\text{XI}:2} - 0.5871 V_{\text{XI}:5}$ $+0.3184\overline{V}_{\text{XI}:\text{6}}-0.0297$	-81
$V_{\rm VII\;:\;4}=-0\cdot 5674V_{\rm VI\;:\;1}-0\cdot 4059V_{\rm VI\;:\;2}$ $+0.3430V_{\rm VI}$: $1+0.3743V_{\rm VI}$: 4 $+0.0322.$.73	$V_{\text{XII}:5} = -0.4179 V_{\text{XI}:4} - 0.3681 V_{\text{XI}:6}$ $+0.0252$	- 56

TABLE 7

Regression equations

NOTE $-V_{I,J}$ represents the J-th C. P. relating to the I-th month.

ORTHOGONAL FIELDS OF TEMPLRATURE VARIATION

Orthogonal space fields of mean temperature over Peninsular India

second C. P. is also correlated by 1 to 3 C. Ps of the previous month.

If P_r is the proportion of the variance accounted by the rth C. P. in the particular month and P' . is the fraction of the variance accounted by the regression equation the fraction of the total variance accounted by the final estimate of the temperature distribution is $\sum P_r P'_r$. The percentage of variance accounted by the several C.Ps of the previous month for estimating the mean temperature of the current month are given against I in Table 8.

Inter-correlations between mean temperatures of successive months have been found to be low generally. The variance accounted by linear regression equations between temperatures of successive months are given for a few stations in Table 8. It is clear that by resolving the tem-

peratures into empirical orthogonal functions we are able to derive useful relationships as compared to the ordinary C.Cs from the original series.

4.3. Seasonal variation of the characteristic patterns For studying the seasonal variation of the characteristic patterns, the matrix made up of the twelve corresponding eigen vectors relating to the different months is considered. The monthly-march of the pattern of spatial distribution is studied by transposing this matrix as before into a fresh set of vectors which are mutually orthogonal and retaining between themselves the total variance in the matrix. The 'eigen vectors' with elements corresponding to the different months of the year indicate the several components of the seasonal variation.

The twelve eigen values and the percentage of variance, they represent, are given in Table 9 and

TABLE 8

Percentage variance accounted by previous month

TABLE 9

Eigen values of the first three C.Ps

TABLE 10

Normalised predominant eigen vectors

324

Orthogonal space fields of mean temperature over Pensulai India

the normalised elements of the predominant eigen vector* corresponding to the first three C.Ps are given in Table 10.

First C. P. - The first eigen vector accounts for 85 per cent of the variance, the second and third accounting for 6 and 5 per cent respectively. The magnitude of the twelve elements of the first eigen vector can be taken as the intensity of the fiirst C. P. which represents largely the influence of continentality on the temperature distribution over the area as a whole. It is seen that this measure attains low values in January and September while in April and November they are highest, thus

indicating that the continentality over the Peninsula is lowest at the end of southwest and northesast monsoon seasons.

Second C. P. - The predominant eigen vector accounts for 53.5 per cent of the variance. The magnitude of the elements of this vector fluctuates considerably from month to month and no regular seasonal variation could be identified.

Third C. P. - The predominat eigen vectors accounts for 26.5 per cent of the variance. Here as well the magnitude of the elements fluctuates considerably.

^{*}In this section we shall refer to the seasonal patterns thus derived as 'eigen vectors' only to avoid confusion with the 'characteristic patterns' referred to in 4.1 and 4.2.

 ${\it REFERENCE}$

Grimmer, M. Haurwitz and Craig $\rm Jagannathan,$ $\rm P_{\bullet}$

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