Effect of low velocity channel on *SH* wave

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(*Received 11 June 1968*)

ABSTRACT. The propagation of *SH* wave was investigated using the velocity distribution function compatible with the actual distribution of shear wave velocity inside the earth. The constants involving the velocity distribution function were calculated from Gutenberg-Birch model and the results discussed.

1. Introduction

The distribution of velocity of seismic body waves inside the earth is known today with good amount of certainty. In spite of some differences of opinion regarding some small zones of the earth, specially the 'mantle basement' and 'innercore', this difference has not affected the interpretation of teleseisms.

In this field Gutenberg and Jeffreys independently have done pioneering work and the travel time curves have now arrived at a reliable velocity distribution of seismic waves inside the earth. The results of these two authors, however, differ slightly with regard to the region that ranges from a depth of about 40 km, *i.e.*, immediately below the Mohorovicic discontinuity, to a depth of about 350 km in the upper mantle.

It has been observed that in the case of earthquakes of crustal origin, the observed amplitudes of the body waves are markedly reduced at epicentral distances ranging between 5 to 15 degrees (Gutenberg 1926, 1959b). In order to explain this, Gutenberg (1926) postulated a low velocity channel in the upper mantle, known as the 'asthenosphere'. This was also earlier pointed out by Barrel (1914) from his work on isostasy.

The validity of this hypothesis was further supported by Gutenberg (1959 a) during the last 20 years. Many authors have supported the view that there is a decrease with depth in the velocity of *P* and *S* waves immediately below the Mohorovicic discontinuity. The velocity reaches its minimum at a depth of about 140 km for transverse waves and a bit less for the longitudinal waves. Caloi (1967) supported this hypothesis on the basis of the existence of channel waves *Pa* and *Sa* propagated in the asthenosphere. The mechanism of propagation of the waves in a channel and their arrival on the earth's surface is different from that of the normal body waves. Body waves appear on seismograms as impulses and often consist of a single impulse only, whereas channel waves appear to be of wave trains due to their propagation through different layers in which seismic wave velocities vary.

Alterman, Jarosch and Pekeris (1962) while studying the observational data of the mantle for the propagation of Rayleigh waves have supported the existence of a low velocity channel in the upper mantle. Takeuchi, Press and Kabayashi (1959) also showed the validity of the above conclusion from the studies of surface waves.

In the present study the authors have made an attempt to calculate the displacement of *SH* waves at various distances on the earth's surface on the assumption of a velocity variation of shear waves at different depths. The Gutenberg-Birch model (Anderson 1964) has been used in calculating the values of the constants occurring in the velocity distribution function.

2. Basic equation and solution of the problem

Let us consider a source on the free surface of a semi-infinite elastic medium producing a disturbance which is symmetrical about z-axis. The positive direction of z-axis is vertically downwards into the medium and is perpendicular to the free surface. Using cylindrical coordinates r, θ, z and applying the symmetry about the axis, the equation of motion for SH wave can be written as—

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} - \frac{U}{r^2} + \frac{\partial^2 U}{\partial z^2} = \frac{1}{\beta^2}\frac{\partial^2 U}{\partial t^2} \tag{1}$$

where U is the displacement, β the velocity of the S-wave.

Let us assume the displacement to be in S.H.M. with frequency, σ, *i.e.*,

$$U(z, r, t) = u(z, r)\, e^{i\sigma t}$$

Equation (1) then reduces to

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{\sigma^2}{\beta^2}u \tag{2}$$

Applying Hankel transformation to the equation (2) and using the well known relations

$$\frac{d}{dx}J_\nu(x) = \frac{\nu}{x}J_\nu(x) - J_{\nu+1}(x)$$

$$\frac{d}{dx}J_{\nu+1}(x) = J_\nu(x) - \frac{\nu+1}{x}J_{\nu+1}(x)$$

we get

$$\frac{d^2}{dz^2}\bar{u}(z, \xi) = \left(\xi^2 - \frac{\sigma^2}{\beta^2}\right)\bar{u}(z, \xi) \tag{3}$$

where $\bar{u}(z, \xi) = \int_0^\infty u(z, r)\, J_1(\xi r)\, r dr$

and $u(z, r) = \int_0^\infty \bar{u}(z, \xi)\, J_1(\xi r)\, \xi\, d\xi$

Let us assume

$$\frac{1}{\beta^2} = \mu_0 - \mu_1\left(1 - \frac{z}{h}\right)^2 + \mu_2 e^{-pz} \quad \text{, for } 0 \leqslant z \leqslant h$$

$$= \mu_0 - \frac{\mu_1}{3} - \mu_2\frac{e^{-ph}-1}{ph} - 2\sum_{n=1}^{\infty}\left[\frac{\mu_1}{n^2\pi^2} + \frac{\mu_2 ph}{p^2h^2+n^2\pi^2}\left\{(-1)^n e^{-ph} - 1\right\}\right]\cos\frac{n\pi z}{h} \quad \text{for } 0 \leqslant z \leqslant h$$

$$= \mu_0 - \frac{\mu_1}{3} - \mu_2\frac{e^{-ph}-1}{ph} - 2\sum_{n=1}^{\infty}\left[\frac{\mu_1}{n^2\pi^2} + \frac{\mu_2 ph}{p^2h^2+n^2\pi^2}\left\{(-1)^n e^{-ph} - 1\right\}\right]\cos\frac{n\pi z}{h} \quad \text{for } z > h \tag{4}$$

Here μ_0, μ_1 and μ_2 are constants and depend on the velocity variation of shear wave. Substituting (4) in (3) we get,

$$\frac{d^2\bar{u}(z, \xi)}{dz^2} = \left[\xi^2 - \sigma^2\left(\mu_0 - \frac{\mu_1}{3} - \mu_2\frac{e^{-ph}-1}{ph}\right) + \right.$$
$$\left. + 2\sum_{n=1}^{\infty}\sigma^2\left\{\frac{\mu_1^2}{n^2\pi^2} + \frac{\mu_2 ph}{p^2h^2+n^2\pi^2}\left[(-1)^n e^{-ph} - 1\right]\right\}\cos\frac{n\pi z}{h}\right]\bar{u}(z, \xi)$$

Substituting $\pi z/h = 2x$

and $\bar{u}(z, \xi) = \bar{u}\left(\frac{2h}{\pi} x, \xi\right) = f(x, \xi)$

We have,

$$\frac{d^2}{dx^2} f(x, \xi) = \left[\frac{2h}{\pi} \left\{\xi^2 - \sigma^2\left(\mu_0 - \frac{\mu_1}{3} - \mu_2\frac{e^{-ph} - 1}{ph}\right)\right\} + \right.$$

$$\left. + \; 2\mu_1 \sum_{n=1}^{\infty}\left(\frac{2h}{\pi}\right)^2 \sigma^2 \left\{\frac{1}{n^2\pi^2} + \frac{\mu_2}{\mu_1} \frac{ph}{p^2h^2 + n^2\pi^2} \left[(-1)^n e^{-ph} - 1 \right] \cos 2nx \right.\right] f(x, \xi) \quad (5)$$

Now equation (5) can be expressed in the form

$$\frac{d^2 f(x)}{dx^2} + Q(x)f(x) = 0 \quad (6)$$

where $Q(x) = \omega^2 + 2\theta \sum_{n=1}^{\infty} \gamma_n \cos 2nx$

Equation (6) represents the Hill's equation and by Floquets theorem (Magnus and Winkler 1966) this equation has the solution

$$f(x) = Ae^{i\alpha x} p_1(x) + Be^{-i\alpha x} p_2(x)$$

where $p_1(x)$ and $p_2(x)$ are periodic functions with period π.

Now defining functions w_1 and w_2 as

$$w_1 = -\omega y + iy', \quad w_2 = -\omega y - iy'$$

equation (6) is equivalent to a system of linear differential equations

$$\frac{\omega}{i} \frac{dw_1}{dx} = \omega^2 w_1 + \theta \sum_{n=1}^{\infty} (\gamma_n \cos 2nx) (w_1 + w_2)$$

$$-\frac{\omega}{i} \frac{dw_2}{dx} = \omega^2 w_1 + \theta \sum_{n=1}^{\infty} (\gamma_n \cos 2nx) (w_1 + w_2)$$

Golomb (1958) (*c.f.* Magnus and Winkler 1966) has shown that α can be determined from the equation

$$\alpha^2 = \omega^2 + \theta^2 \sum_{n=-\infty}^{\infty} \frac{\gamma_{|n|}}{(2n + \alpha)^2 - \omega^2} +$$

$$+ \theta^3 \sum_{\substack{n, m = -\infty \\ n \neq m}}^{\infty} \frac{\gamma_{|n|} \gamma_{|m|} \gamma_{|n-m|}}{[(2n + \alpha)^2 - \omega^2] [(2m + \alpha)^2 - \omega^2]} + O(\theta^4)$$

(where $\gamma_0 = 0$), provided that θ is sufficiently small.

Thus α may be written in the form (with $\gamma_0 = 0$)

$$\alpha = \pm \left[\omega + \frac{\theta^2}{4\omega} \sum_{n=1}^{\infty} \frac{\gamma_n^2}{n^2 - \omega^2} + \frac{\theta^3}{32\omega} \sum_{\substack{n, m = -\infty \\ n \neq m}}^{\infty} \frac{\gamma_{|n|} \gamma_{|m|} \gamma_{|n-m|}}{nm (n + \omega) (m + \omega)} + O(\theta^4) \right]$$

In the present case we have

$$\omega^2 = -\left(\frac{2h}{\pi}\right)\left\{\xi^2 - \sigma^2\left(\mu_0 - \frac{\mu_1}{3} - \mu_2\frac{e^{-ph}-1}{ph}\right)\right\}$$

$$\theta = \mu_1$$

$$\gamma_n = -\left(\frac{2h}{\pi}\right)^2 \sigma^2\left\{\frac{1}{\pi^2 n^2} + \frac{\mu_2}{\mu_1}\frac{ph}{p^2h^2+\pi^2n^2}\left[(-1)^n e^{-ph}-1\right]\right\} \quad (7)$$

Since μ_1 is small, solution (7) can be applied in the present case.

Let us choose $p_1(x) = p_2(x) = \cos 2x$

so that

$$f(x,\xi) = [Ae^{i\alpha x} + Be^{-i\alpha x}] \text{ os } 2x \quad (7a)$$

where A and B are constants and can be determined from the boundary conditions.

The boundary condition on the surface is

$$T_{z\theta}\Big|_{z=0} = \mu S\frac{1}{r} \quad (8)$$

Or,
$$\frac{du(z,r)}{dz}\Big|_{z=0} = \frac{S}{r}$$

Or,
$$\frac{d\bar{u}(z,\xi)}{dz}\Big|_{z=0} = \frac{S}{\xi}$$

Or,
$$\frac{df(x,\xi)}{dx} = \frac{2hS}{\pi\xi} \quad (9)$$

where S is a constant depending on the source.

Now when α is real, equation (7a) becomes

$$f(x,\xi) = [(A+B)\cos\alpha x + i(A-B)\sin\alpha x]\cos 2x$$

The condition in (9) shows that in this case the constants A and B are imaginary, and hence to make the displacement to be real we require

$$A + B = 0$$

Hence, we get

$$f(x,\xi) = 2iA\sin\alpha x\cos 2x$$

We may note that this displacement remains finite as $z \to \infty$ or $x \to \infty$

And when α is positive imaginary, equation (7a) shows that for the displacement to be finite at infinity we require $B = 0$ and hence

$$f(x,\xi) = Ae^{i\alpha x}\cos 2x$$

Now using the condition in equation (9) we get

$$2A = \frac{2hS}{\pi i\alpha\xi}, \text{ when } \alpha \text{ is real;}$$

$$\text{and } A = \frac{2hS}{\pi i\alpha\xi}, \text{ when } \alpha \text{ is imaginary.}$$

Therefore,
$$f(x,\xi) = \frac{2hS}{\pi\alpha\xi}\sin\alpha x\cos 2x, \text{ when } \alpha \text{ is real}$$

$$= \frac{2hS}{\pi i\alpha\xi}e^{i\alpha x}\cos 2x, \text{ when } \alpha \text{ is imaginary.} \quad (10)$$

In the present case

$$\alpha = \omega + \frac{\mu_1^2}{4\omega} \sum_{n=1}^{\infty} \frac{\gamma_n^2}{n^2 - \omega^2} + \cdots\cdots\cdots$$

where ω and γ_n are given by Eq. (8).

Neglecting μ_1^2 and other higher orders, we have

$$\alpha \simeq \omega = \frac{2h}{\pi} \sqrt{a^2 - \xi^2}$$

where $$a^2 = \sigma^2 \left(\mu_0 - \frac{\mu_1}{3} - \mu_2 \frac{e^{-ph} - 1}{ph}\right)$$

so that $$\alpha \simeq \frac{2h}{\pi} \sqrt{a^2 - \xi^2} \quad , \text{ for } \xi < a$$

and $$i\,\alpha \simeq -\frac{2h}{\pi} \sqrt{\xi^2 - a^2} \quad , \text{ for } \xi > a$$

Hence from equation (10) we get

$$f(x, \xi) = \frac{S}{\xi \sqrt{a^2 - \xi^2}} \sin\left(\frac{2hx}{\pi} \sqrt{a^2 - \xi^2}\right) \cos 2x\,, \qquad \text{for } \xi < a$$

$$= -\frac{S}{\xi \sqrt{\xi^2 - a^2}}\, e^{-\frac{2hx}{\pi}\sqrt{\xi^2 - a^2}} \cos 2x\,, \qquad \text{for } \xi > a$$

Now applying Hankel inversion, we have

$$u(z, r) = S \cos \frac{\pi z}{h} \int_0^{\infty} \Psi(z, \xi)\, \xi\, J_1(\xi r)\, d\xi \tag{11}$$

where $$\Psi(z, \xi) = \frac{\sin(z \sqrt{a^2 - \xi^2})}{\xi \sqrt{a^2 - \xi^2}}, \qquad \text{for } \xi < a$$

$$= -\frac{\exp(-z \sqrt{\xi^2 - a^2})}{\xi \sqrt{\xi^2 - a^2}}, \qquad \text{for } \xi > a$$

Performing the integration (Erdelyi *et al.* 1959) we get

$$u(z, r) = \frac{S\pi}{2} J_{\frac{1}{2}} \left[\tfrac{1}{2}a \{(z^2 + r^2)^{\frac{1}{2}} - z\}\right] Y_{\frac{1}{2}} \left[\tfrac{1}{2}a \{(z^2 + r^2)^{\frac{1}{2}} - z\}\right] \cos \frac{\pi z}{h} \tag{12}$$

where $J_\nu(\xi)$ and Y_ν are the Bessel function of the first and second kind of order of ν respectively.

For the displacement on free surface $z = 0$ we get

$$u(0, r) = S \int_0^{\infty} \Psi(0, \xi)\, \xi\, J_1(\xi r)\, d\xi \tag{13}$$

where $$\Psi(0, \xi) = 0, \qquad \text{for } \xi < a$$

$$= -\frac{1}{\xi \sqrt{\xi^2 - a^2}}, \qquad \text{for } \xi > a$$

Performing integration in equation (13) we have

$$u(0, r) = \frac{S\pi}{2} J_{\frac{1}{2}}(\tfrac{1}{2}ar)\, Y_{\frac{1}{2}}(\tfrac{1}{2}ar)$$

$$= -\frac{S}{ar} \sin(ar) \tag{14}$$

Now we see from Eq. (12) that $u(z, r) \to u(0, r)$ as $z \to 0$.

Therefore expression of $u(z, r)$ in Eq. (12) is valid when $z \geqslant 0$

Hence the displacement gives as

$$U = u(z,r)\, e^{i\sigma t}$$

$$= \frac{S\pi}{2} J_{\frac{1}{2}}\left[\tfrac{1}{2}a\ \{(z^2+r^2)^{\frac{1}{2}}-z\}\right] Y_{\frac{1}{2}}\left[\tfrac{1}{2}a\ \{(z^2+r^2)^{\frac{1}{2}}-z\}\right] \cos\frac{\pi z}{h}\, e^{i\sigma t} \tag{15}$$

and on $z = 0$

$$U_o = -\frac{S}{ar} e^{i\sigma t} \sin ar$$

$$= -\frac{S}{ar} e^{i\sigma t} \cos\left(\frac{\pi}{2} - ar\right)$$

$$= -\frac{S}{2ar}\left[\exp i\left(\sigma t - ar + \frac{\pi}{2}\right) + \exp i\left(\sigma t + ar - \frac{\pi}{2}\right)\right] \tag{16}$$

Now

$$a^2 = \sigma^2\left(\mu_0 - \frac{\mu_1}{3} - \mu_2\frac{e^{-ph}-1}{ph}\right) \tag{17}$$

$$= \sigma^2 k^2$$

where

$$k^2 = \mu_0 - \frac{\mu_1}{3} - \mu_2\frac{e^{-ph}-1}{ph}$$

Thus

$$a = \sigma k$$

and U_o in Eq. (16) may be written as

$$U_o = -\frac{S}{2ar}\left[\exp\left\{ik\sigma\left(-r+\frac{t}{k}+\frac{\pi}{2k\sigma}\right)\right\} + \exp\left\{ik\sigma\left(r+\frac{t}{k}-\frac{\pi}{2k\sigma}\right)\right\}\right] \tag{18}$$

3. Discussion

Equation (18) shows the displacement corresponding to the velocity of S-wave as prescribed in equation (4). The parameter 'k', which is a function of velocity distribution inside the earth as well as the depth of low velocity channel, may be evaluated on the basis of the known velocity distribution model. To evaluate the constants μ_0, μ_1, μ_2, p and k, the data from the S-wave velocity distribution upto a depth of 140 km prescribed in Gutenberg-Birch model (Anderson 1964) were consulted. This model practically reviews the velocity prescribed by Gutenberg (1959 a, b), Lehmann (1961) and Dorman, Ewing and Oliver (1960). Method of least square was applied and the constants evaluated are as follows:—

$\mu_0 = 0{\cdot}05189$ $\qquad\mu_2 = 0{\cdot}62595$

$\mu_1 = 0{\cdot}01100$ $\qquad p = 0{\cdot}10000$ and

$$k = 0{\cdot}30485 \tag{19}$$

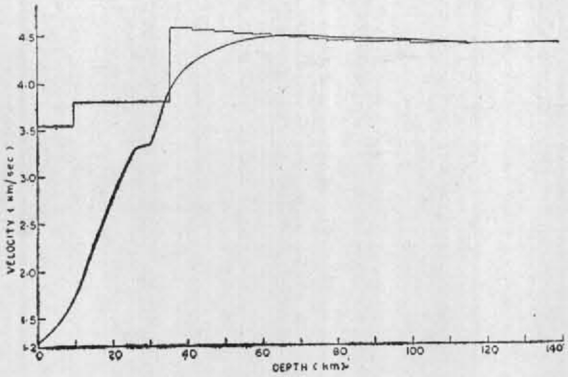

Fig. 1

Discontinuous lines show the velocity distributions of shear wave from Gutenberg–Birch model. Continuous curve shows distribution of shear wave

Having known the values of the constants the velocities of *S*-wave at different depths have been compared graphically in Fig. 1 with that of the actual distribution as prescribed in Gutenberg-Birch model. It is seen from χ^2-test of significance that the fitness of the curve thus drawn has got more than 99% level of significance.

Equation (18) indicates that the amplitude of the displacement is inversely proportional to the distance traversed as observed in general for the body waves (Ewing, Jardetzky and Press 1957). It is also inferred that the amplitude of the displacement is directly proportional to a function of the source energy. It may be noticed that equation (18) which gives a solution of the problem is similar to the solution of the wave motion. The velocity of propagation of shear wave is '$1/k$' which depends on the depth of the low velocity channel and the velocity distribution inside the earth. From the values of the constants given in (19) we get, $1/k = 3{\cdot}2803$ km/sec. It is interesting to note that this velocity is closely agreeing to the velocity of *Lg* (Bath 1956).

4. Acknowledgement

The authors are thankful to Dr. A. N. Tandon, Director, Seismology for guidance and encouragement and to Shri H. M. Chaudury, Seismologist for helpful discussions at many stages of the problem.

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