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Lee waves associated with a large circular mountain

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ABSTRACT. The present work is concerned with the baroclinic properties of stationary waves behind a large circular mountain. It is based on a linear, baroclinic model which includes the Coriolis force f, and its variation with latitude β. The free stream velocity U and the stability parameter l is assumed to be independent of height. A wave equation is derived for the vertical component of perturbation velocity. Its asymptotic solution indicates that the wave amplitude is a function of the non-dimensional parameter ($m=fL/lUd$), where L and d are unit lengths along the horizontal and vertical axes of reference. Numerical values of the solution are presented to indicate the attenuation of the wave with radial distance, with variations in m and with height.

1. Introduction

The theoretical and practical aspects of the mountain wave problem have received considerable attention in the last decade. Important theoretical work on stationary lee waves is available in the investigations of Stewart (1948), Charney and Eliassen (1949), Bolin (1950), Smagorinsky (1953), Kurbatkin (1959) and Saltzman (1963). The first three investigations were concerned with an essentially barotropic atmosphere. Baroclinic features were introduced by Smagorinsky and, recently, a very comprehensive work on meridional motion created by heat sources and orographic barriers is available in the paper by Saltzman (1963). Kurbatkin (1959) also considered baroclinic features, but assumed a rigid upper lid at the top of the atmosphere. In the present investigation, a few results are presented on the baroclinic disturbance in the lee of a large circular mountain, without the assumption of a rigid lid.

If we consider, as a first approximation, a homogeneous air stream, without shear and with constant static stability, then physical considerations enable us to distinguish between three horizontal scales of interest in the mountain wave problem (Queney *et al.* 1960). They are,

(*a*) 10 km : Non-hydrostatic and negligible Coriolis force;

(*b*) 100 km: Hydrostatic and non-geostrophic, with constant Coriolis force ; and

(*c*) 1000 km: Hydrostatic and geostrophic with a variable Coriolis force.

A large barrier, such as the Tibetan Plateau, comes under category (*c*).

Apart from considerations of scale, the flow pattern behind a mountain depends on the choice of appropriate boundary conditions. In the present study we have assumed a linear model. This restricts the scope of the investigation to mountains of small height.

By linearizing the lower boundary condition, we assume that the flow is entirely over the mountain, that is, there is no cross-mountain flow at the lower boundary. This is a departure from reality, but is an acceptable approximation for mountains less than one kilometre in height (Charney and Drazin 1961, Phillips 1963).

Physical considerations require that the vertical component of the perturbation be bounded at the upper boundary. But, as is well known, the upper and lower boundary conditions by themselves are not sufficient to give a unique solution to the problem ; an additional constraint is required to make the problem determinate. In this investigation, we make use of the radiation condition for outgoing waves at infinity.

The principal assumptions may be summarized as follows :

(i) We consider a hypothetical circular mountain extending up to 1000 km in a radial direction, but of 1 km height. The computations assume a flat earth on which the β plane approximation is valid.

(ii) At the lower boundary the flow is assumed to be entirely over the mountain.

(iii) The free stream velocity U is assumed to be representative of a steady zonal current without shear. We also assume that the static stability of the atmosphere is invariant with height.

(iv) The effect of turbulence and frictional dissipation is neglected.

(v) Non-linear effects are not considered in the present work.

2. Basic Equations

Let U represent the freestream velocity and let $\bar{p}$, $\bar{\rho}$, $\bar{\theta}$ be the pressure, density and potential temperature of the undisturbed air. In Cartesian co-ordinates (ox, oy, oz) the perturbation velocity components are specified by u, v and w, while p, ρ and θ stand for perturbations of pressure, density and potential temperature. If we represent the perturbation vorticity ζ and divergence D by

$$\zeta = (\partial v/\partial x) - (\partial u/\partial y), \quad (2\cdot1)$$

$$D = (\partial u/\partial x) + (\partial v/\partial y), \quad (2\cdot2)$$

then, for steady motion, the change in vorticity and divergence is given by

$$U(\partial\zeta/\partial x) + \beta v = -fD \quad (2\cdot3)$$

$$U(\partial D/\partial x) - f\zeta + \beta u = -(1/\bar{\rho})\nabla_1^2 p, \quad (2\cdot4)$$

where $f = 2\,\Omega \sin\phi$, $\beta = df/dy$ and ∇_1^2 is the Laplacian operator in two dimensions. We have omitted non-linear terms in (2.3) and (2.4).

On eliminating ζ and using the geostrophic relation we get

$$D_{xx} + (f/U)^2 D = -(1/\bar{\rho}U)\,[\nabla_1^2 + (\beta/U) - (\beta/f)(\partial/\partial y)]\,p_x, \quad (2\cdot5)$$

where subscripts have been used to denote derivatives.

To compensate for compressibility, it is convenient to change over to the co-ordinates of Palm and Foldvik (1959). We have

$$u,\,v,\,w = [\bar{\rho}(0)/\bar{\rho}(z)]^{\frac{1}{2}}\,(u',\,v',\,w'), \quad (2\cdot6)$$

$$p = [\bar{\rho}(z)/\bar{\rho}(0)]^{\frac{1}{2}}\,p'. \quad (2\cdot7)$$

The subscripts 0, z refer to the earth's surface and the vertical co-ordinate of the point under consideration. For simplicity in notation, we shall hereafter drop the primes with the understanding that we are dealing with variables defined by (2·6) and (2·7). It is important to note that when we introduce this transformation $\bar{\rho}(z)$ on the right hand side of (2·5) is replaced by the constant $\rho(0)$. As we shall see later, this constant is eventually eliminated in the final equation for w.

We express the equation of continuity and the first law of thermodynamics by the relations

$$D = -w_z - \lambda w, \quad (2\cdot8)$$

$$w_{xx} + l^2 w = -\frac{1}{\rho(0)\,U}\,(p_{xz} + \lambda p_x), \quad (2\cdot9)$$

where $\lambda = -\frac{1}{2}(\partial/\partial z)\ln\bar{\rho}(z) - g/c^2$,

$l^2 = gB/U^2$,

$B = \partial/\partial z \ln\bar{\theta}$ and

c is the velocity of sound.

In (2·8) we have assumed that

$$1/\bar{\rho}\,(d\bar{\rho}/dt) \approx -(g/c^2)w.$$

It is to be noted that (2·9) results from a combination of the first law of thermodynamics and the vertical equation of motion.

In the latter equation we have retained the vertical acceleration, but not the Coriolis force. The omission of the vertical Coriolis force is justified for the scale of motion in which we are interested. It could be argued on similar grounds that, for this scale of motion, it is hardly necessary to include the vertical acceleration. We have, however, retained this term to show that the final equation may be reduced, if necessary, to the one used for defining the flow past a small three-dimensional plateau, or a small two-dimensional mountain. In the subsequent analysis, the retention of this term is of no consequence.

On eliminating D and p from (2·5), (2·8) and (2·9) we finally get

$$L(w) = 0, \qquad (2\cdot10)$$

where

$$L \equiv (\partial^2/\partial x^2)(\nabla^2)+(f/U)^2\,\partial^2/\partial z^2+l^2\nabla_1^2+ +\beta/U(\partial^2/\partial x^2+l^2)-\lambda^2(\partial^2/\partial x^2+f^2/U^2)- -\beta/f[(\partial^2/\partial x^2)+l^2]. \qquad (2\cdot11)$$

There are two interesting cases in our consideration of (2·11)—

(*i*) If we ignore terms containing f, β and λ, we get

$$[(\partial^2/\partial x^2)(\nabla^2)+l^2\nabla_1^2]\ w = 0.$$

This equation was solved by Wurtele (1957) for a small three-dimensional plateau.

(*ii*) If we also omit the derivatives with respect to y, we are left with

$$w_{xx}+w_{zz}+l^2 w = 0.$$

This is the basic equation for a two-dimensional mountain on which extensive investigations have been carried out in the past (Scorer 1949, Crapper 1959, 1962).

Our aim is to solve (2·10) with the terms containing f and β, but considerable simplification is possible on considering the order of magnitude of different terms. Let us consider the following representative values ;

$$c = 3\times10^{-1}\,\text{km sec}^{-1}$$

$$g = 10^{-2}\,\text{km sec}^{-2}$$

$$f = 10^{-4}\ \text{sec}^{-1}$$

$$U = 10^{-2}\,\text{km sec}^{-1}$$

$$l^2 = 1\cdot0\ \text{km}^{-2}$$

$$\lambda^2 = 10^{-3}\,\text{km}^{-2} \text{ and}$$

$$\beta = 1\cdot6\times10^{-8}\ \text{km}^{-1}\ \text{sec}^{-1}$$

Using the equation of state, we note that

$$\lambda = -g/c^2\,[1-\gamma/2(1+R/g\ \partial\bar{T}/\partial z)],$$

where $\gamma = c_p/c_v$. Hence, for an isothermal atmosphere

$$\lambda = -0\cdot3\ g/c^2$$

and with $\partial\bar{T}/\partial z = -\cdot006°c/m$,

$$\lambda = -0\cdot4\ g/c^2.$$

Consequently, λ^2 may be taken to be of the order of 10^{-3} km^{-2} in our computations.

As the horizontal extent of the mountain is 1000 km, we take

$$\partial/\partial x \sim \partial/\partial y \sim 10^{-3}\,\text{km}$$

and $\partial/\partial z \sim 10^{-1}\ \text{km}^{-1}$.

The nine operators and multipliers in (2·11) now have the following orders of magnitude in units of km^{-4} :

(1) $\partial^4/\partial x^2\partial z^2 \sim 10^{-8}$

(2) $\partial^2/\partial x^2(\nabla_1^2) \sim 10^{-12}$

(3) $(f/U)^2\ (\partial^2/\partial z^2) \sim 10^{-6}$

(4) $l^2\nabla_1^2 \sim 10^{-6}$

(5) $(\beta/U)(\partial^2/\partial x^2) \sim 10^{-12}$

(6) $\beta l^2/U \sim 10^{-6}$

(7) $\lambda^2(\partial^2/\partial x^2+f^2/U^2) \sim 10^{-7}$

(8) $(\beta/f)(\partial/\partial y)(\partial^2/\partial x^2) \sim 10^{-13}$

(9) $(\beta l^2/f)(\partial/\partial y) \sim 10^{-7}$

We note that the Rossby number($R_0=U/fL$) for flow over a mountain of this size is of the order of 0·10. For the purpose of the present investigation, we may only retain the three largest terms, namely, those which have an order of magnitude of 10^{-6} km^{-4}. Equation (2·10) is then simplified to

$$l^2(w_{xx}+w_{yy})+(f/U)^2 w_{zz}+(\beta l^2/U)w=0 \quad (2\cdot12)$$

For convenience in computation, it is desirable to express (2·12) in non-dimensional units. Let L, d be two representative lengths, such that L is equal to the maximum horizontal extent of the mountain (1000 km) and d is approximately the height of the tropopause (10 km).
Putting

$$x^*=x/L,\ y^*=y/L,\ z^*=z/d \quad (2\cdot13)$$

in (2·12), we find

$$\nabla_1^2 w+m^2 w_{zz}+k^2 w=0, \quad (2\cdot14)$$

where $m^2=(fL/lUd)^2$,
$k^2=\beta L^2/U$. (2·15)

To simplify the notation we have omitted starred symbols, but it will be understood that we refer to non-dimensional units defined by (2·13).

The effect of "baroclinicity", or static stability which renders the perturbations baroclinic, is contained in the factor (fL/lUd), which occurs as the coefficient of w_{zz} in (2·14). This factor has an interesting physical interpretation.

If it is put equal to unity, then (2·14) is the appropriate equation for only barotropic perturbations (Stewart 1948). But, as we can see, one can always put the coefficient (fL/lUd) equal to unity for any static stability by choosing the proper ratio of the horizontal to the vertical unit length, that is, by choosing a suitable value of (L/d).

The non-dimensional parameter $(m=fL/lUd)$, can be also expressed by a combination of the Froude number $(F=U^2/gd)$ with the Rossby number $(R_0=U/fL)$. We have

$$m=(F/R_0)\times(g/lU^2). \quad (2\cdot16)$$

In the present investigation (F/R_0) is fixed by the dimensions of the mountain and the free stream velocity U. Consequently, it is possible to study the distortion of the perturbation as a function of the parameter m by varying the static stability l.

3. Method of solution

Let us define a stream function ψ where

$$\psi=\int(w/U)\,dx \quad (3\cdot1)$$

then w may be replaced by ψ in (2·14), because any solution of (2·14) will also hold for ψ defined by (3·1).

For obtaining numerical values of the solution, it is convenient to use a theorem of Hsu (1948) for inverting a double Fourier transform. This theorem was also used by Wurtele (1957) for three-dimensional flow over a plateau.

Let

$$\phi=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi(x,y,z)\exp-i(sx+ty)dx\,dy \quad (3\cdot2)$$

From (2·14) we have

$$(d^2\phi/dz^2)+\mu^2\phi=0, \quad (3\cdot3)$$

where $\mu^2=[k^2-(s^2+t^2)]/m^2$. (3·4)

The appropriate solutions of (3·3) are

$$\phi=A\exp-\mu z,\ A\exp i\mu z \quad (3\cdot5)$$

depending on whether μ is complex or real. We have chosen the positive complex solution to represent outgoing waves at infinity. When μ is real, the solution with negative μ is the only realistic one possible.

The constant A is fixed by the shape of the mountain

Let us consider the following mountain,

$$\psi_0=H\exp-a^2(x^2+y^2), \quad (3\cdot6)$$

whence

$$\phi_0=\pi H/4a^2\exp-(s^2+t^2)/4a^2, \quad (3\cdot7)$$

where subscript 0 refers to $z = 0$.

Introducing the polar co-ordinates

$$s=p\cos\alpha,\qquad x=r\cos\theta,$$
$$t=p\sin\alpha,\qquad y=r\sin\theta, \tag{3·8}$$

we obtain

$$4\pi^2\psi = \int_{-\infty}^{\infty}\int_{-\pi/2}^{\pi/2}\phi\times\exp ipr\cos(\alpha-\theta)\times p\,dp\,d\theta \tag{3·9}$$

From the meteorological point of view, the second solution of (3·5) is more interesting, because it represents waves which can propagate energy to great heights. The first solution only represents waves which are rapidly damped with height. We may, therefore, use the second solution and try to evaluate the asymptotic form of (3·9). We have

$$\psi = H/16\pi a^2\int_{-\infty}^{\infty}\int_{-\pi/2}^{\pi/2}\exp -p^2/4a^2\times\exp i\times$$
$$\times\,[(k^2-p^2)^{\frac{1}{2}}z/m+pr\cos(\alpha-\theta)]\times p\,dp\,d\theta. \tag{3·10}$$

To find the asymptotic form of the above integral, we evaluate the saddle point (p_0, α_0) of the expression—

$$h(p,\alpha)=(k^2-p^2)^{\frac{1}{2}}z/m+pr\cos(\alpha-\theta). \tag{3.11}$$

If we omit the negative saddle point from the path of integration to eliminate the upstream wave, we get

$$p_0 = mkr/R,\ \tan\alpha_0 = y/x, \tag{3·12 a,b}$$

where

$$r^2=x^2+y^2,\ R^2=m^2r^2+z^2. \tag{3.13 a,b}$$

Noting that

$$\left[h_{pp}\,h_{\alpha\alpha}-h^2_{\alpha p}\right]^{\frac{1}{2}}_{p_0,\ \alpha_0}=Rr/z,$$
$$h(p_0,\alpha_0)=kR/m,$$

we arrive at the following expression for ψ

$$\psi\sim H/16\pi a^2\times\exp-(mkr/2aR)^2\times$$
$$\times\,mkz/R^2\times\exp(ikR/m). \tag{3·14}$$

It is of interest to note that the asymptotic solution (3·14) contains the factor $\exp(ikR/m)/R$, which we know is the fundamental solution for outgoing waves at infinity.

4. Numerical results

Numerical values of $\psi(r,\theta,z)$ may be obtained from (3·14). For this purpose we may only consider the real part of ψ, so that the last term of (3·14) is replaced by $\cos(kR/m)/R$, which is available in a tabulated form.

Let us consider the following numerical values of H and a

$$H = 0·1,$$
$$a = 2·0. \tag{4·1}$$

The maximum height of the mountain is then 1·0 km, and H is reduced to one hundredth of its maximum value at about 1000 km from the centre of the mountain.

If we substitute in (2·15) the numerical values of f, L, U and d, as given in section 2, we find

$$m = 1/l. \tag{4·2}$$

We have, therefore, evaluated ψ as a function of m by considering different values of l. The results are shown in Table 1, where $z=0·5$ and the different values of m correspond to l = 3, 2, 1 and 0·5.

The variation of ψ with m is shown graphically in Fig. 1. The interesting features are (a) an increase in wave amplitude with decrease in m and (b) the relatively small change in wave length with variations in m. The first result would indicate that with large values of the static stability l, m is small and the wave amplitude is large. But, as we have seen earlier, the interpretation for varying m is much more general, because we could have also decreased m, by altering the ratio of the horizontal and vertical unit lengths L/d.

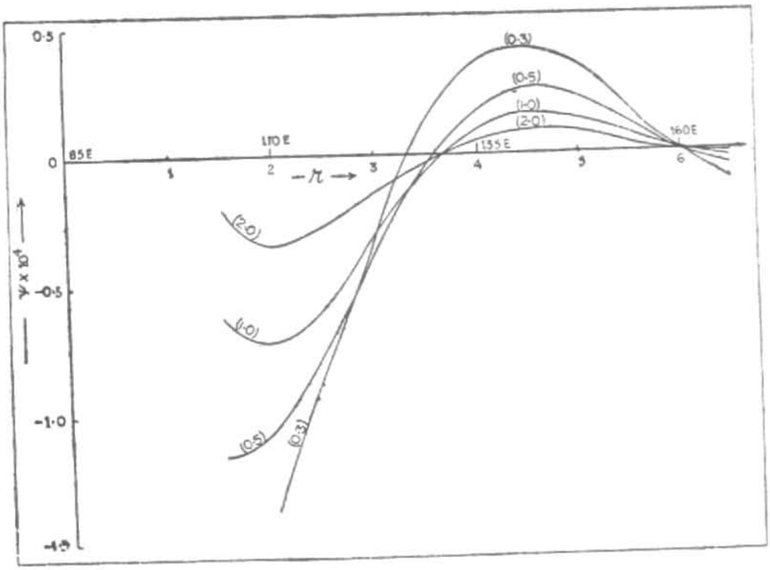

Fig. 1. Variation of ψ with *m*

(Figures in parenthesis represent values of *m*)

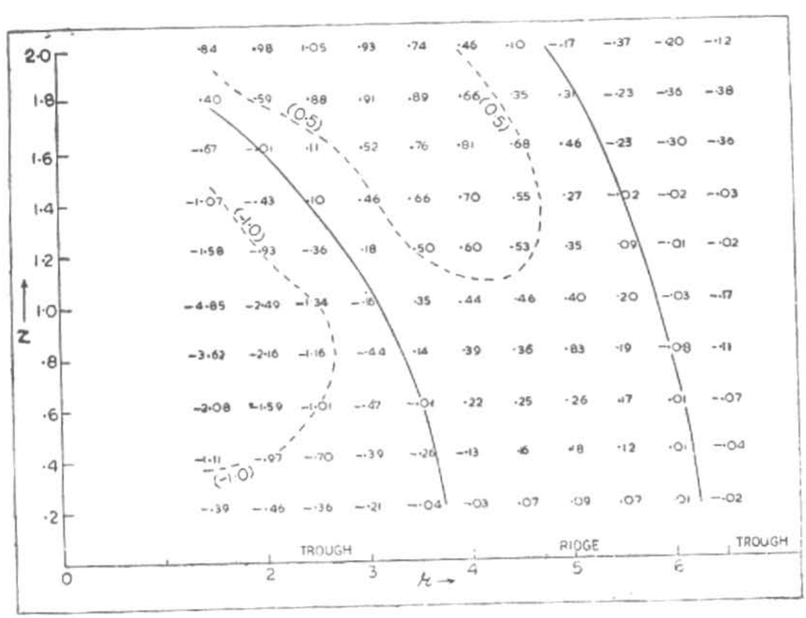

Fig. 2. Variation of ψ with *z*

Figures indicate values of $\psi \times 10^4$ at each grid point. Nodal lines are represented by full lines. Dotted lines represent intermediate contours of $\psi \times 10^4$.

We also computed the variation of ψ with z for one particular value of m (0·5). The results are shown in Fig. 2, where the baroclinic feature of the disturbance is clearly seen. The wave fronts tilt backwards with height, and there is an increase in wave amplitude with altitude.

It is difficult to state how far the computations shown in Fig. 1 agree with observations. There are very few upper air observations available from Tibet and adjoining China. We note, however, that if the centre of the circular mountain is taken to be along the meridian 85°E and between 30°—40°N, then the first major trough line is between 105°E and 110°E. This is in reasonable agreement with the few observations we have of the quasi-stationary trough observed to the east of Tibet in winter.

There is another significant difference between the present work and Wurtele's earlier investigation (1957). In Wurtele's work, the nodal lines (isopleths of w=0) are rectangular hyperbolae. This gives a crescent-shaped region of updraft behind the barrier. But, in our work, we see from (3·14) that the nodal lines are represented by

$$kR/m = (2n+1)\,\pi/2. \qquad (4\cdot3)$$

Hence, if $x = x_0$, $y = 0$ are taken as reference points, then the nodal lines in the xy-plane are the circles

$$(x/x_0)^2 + (y/x_0)^2 = 1. \qquad (4\cdot4)$$

If we refer to the work of Kurbatkin (*loc. cit*), who also found a crescent-shaped region of updraft, we find that this is a consequence of non-geostrophic motion. The inclusion of the first term in (2·11) ultimately yields a wave equation, whose solution is only specified within its characteristic cone. In our treatment the first term of (2·11) is at least two orders of magnitude smaller than the terms which have been retained. Consequently, we do not find a crescent-shaped region of updraft.

TABLE 1

Variation of ψ with m (z = 0·5)

r	m = 0·3	0·5	1·0	2·0
		$\psi \times 10^4$		
1·5	—1·81	—1·17	—·62	—·20
2·0	—1·39	—1·07	—·73	—·35
2·5	— ·76	— ·74	—·58	—·24
3·0	— ·33	— ·38	—·32	—·16
3·5	·12	— ·06	—·07	—·04
4·0	·35	·15	·08	·05
4·5	·40	·25	·15	·08
5·0	·35	·22	·13	·08
5·5	·17	·12	·11	·03
6·0	·01	·02	·02	·01
6·5	— ·12	—·06	—·03	—·01

5. Summary and conclusions

The principal results of the study may be summarized as follows.

(*i*) For a large circular mountain, extending up to 1000 km in a radial direction, the non-dimensional parameter

$$m = fL\,/\,lUd$$

provides a measure of wave distortion. This is a combination of the Froude Number F, the Rossby Number R_0 and the static stability l.

(*ii*) The wave amplitude increases as we decrease m. With large values of the static stability l, m is small and the wave amplitude is large.

(*iii*) Variations in m appear to have little influence on the wave length of lee waves.

(*iv*) The variation of ψ along the vertical is shown in Fig. 2 for $m = 0\cdot5$. We note that the wave front tilts backwards with height.

(*v*) The nodal lines, in the present work, are circles. When we consider a smaller obstacle, as in Wurtele's investigation (*loc. cit*), the nodal lines are rectangular hyperbolae. This appears to be a consequence of non-geostrophic motion, which is important for a small mountain, but negligible in our work.

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REFERENCES

Bolin, B.	1950	*Tellus*, **2**, 3, p. 184-195.
Charney, J. G. and Eliassen, A.	1949	*Ibid.*, **1**, 2, pp. 38-54.
Charney, J. G. and Drazin, P. G.	1961	*J. geophys. Res.*, **66**, 1, pp. 83-109.
Crapper, G. D.	1959	*J. Fluid. Mech.*, **6**, pp. 51-61.
	1962	*Phil. Trans.*, **254**, A, pp. 601-623.
Hsu, L. C.	1948	*Amer. J. Math.*, **70**, pp. 698-708.
Kurbatkin, G. P.	1959	*Izv. Akad. Nauk.*, U.S.S.R., Ser. Geofiz., 4, 581-592.
Palm, E. and Foldvik, A.	1959	*Geofys. Publ.*, Oslo, **21**, 6, pp. 1-30.
Phillips, N. A.	1963	*Rev. Geophys.*, Amer. geophys. Un., **1**, 2, pp. 123-176.
Queney, P. *et al.*	1960	*W.M.O. Tech. Note*, 34, pp. 1-135.
Saltzman, B.	1963	*J. atmos. Sci.*, **20**, 3, pp. 226-235.
Scorer, R. S.	1949	*Quart. J.R. met. Soc.*, **75**, 323, pp. 41-56.
Smagorinsky, J.	1953	*Ibid.*, **79**, pp. 342-366.
Stewart, H. J.	1948	*J. Met.*, **5**, 5, pp. 236-238.
Wurtele, M. G.	1957	*Beitr. Phys. Atmos.*, **29**, pp. 242-252.