550.342 : 534.647

# Propagation of Love Waves in a medium containing a heterogeneous layer

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(*Received 18 January 1969*)

ABSTRACT. Love wave propagation in a medium consisting of a heterogeneous layer of a finite depth embedded between a homogeneous layer and a half-space has been studied. SH-displacement due to an infinite line source on the free surface has also been investigated.

### 1. Introduction

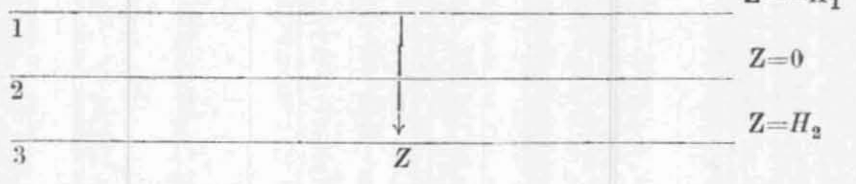
Recent knowledge about the structure of the earth's interior reveals the existence of heterogeneous layers which influence the phase velocities of surface waves. The seismic wave velocity which is dependent on the elastic constant of the material, increases in general with the increase of depth.

Various authors, Meissner (1921), Bateman (1928), Jeffreys (1928), Sato (1952) and Das Gupta (1953) have studied the propagation of surface waves in a heterogeneous medium taking into account different variations of rigidity and density with the increase in depth.

In this paper the propagation of Love waves has been investigated in a medium having a heterogeneous layer embedded between a layer of finite depth and a half-space, both being homogeneous.

### 2. Basic equation and solution

Let us consider a vertically heterogeneous layer of finite thickness embedded between a homogeneous layer lying above it and a homogeneous half-space below. Let us also consider that a disturbance is created by a periodic line source on the free surface $Z=-H_1$, along the Y axis. The Z axis is taken vertically downwards while the X axis is along the direction of propagation. The origin is taken on the plane $Z=0$ separating the first and the second layers. Let $Z=H_2$ be the plane of interface between the second layer and the half-space.




Let the rigidity and density of the first layer and the half-space be given by $\mu_1, \rho_1$ and $\mu_3, \rho_3$ and for the second layer,

$$\mu_2 = \bar{\mu} \cosh (pz - \lambda), \quad \beta_2^2 = (\bar{\mu}/\bar{\rho}) \cosh^2 (pz - \lambda)$$

where $\beta_2$ is the S-wave velocity in the second layer.

The equation of motion for the heterogeneous second layer can be written as (Ewing, Jerdetzky and Press 1957) —

$$\rho_z \frac{\partial^2 v}{\partial t^2} = \mu_2 \nabla^2 v + \frac{\partial \mu_2}{\partial z} \cdot \frac{\partial v}{\partial z} \tag{1}$$

where $v(x, z)$ is the displacement and let

$$v = \phi_2(z)\cos(\omega t - kx) \tag{2}$$

then, $\phi_2$ will be a solution of the equation—

$$\frac{d^2\phi_2}{dz^2} + \left(\frac{\omega^2}{\beta_2^2} - k^2\right)\phi_2 + \frac{1}{\mu_2}\,\frac{d\mu_2}{dz}.\,\frac{d\phi_2}{dz} = 0 \tag{3}$$

on substitution $\bar{\phi}_2 = \sqrt{\mu_2}\,\phi_2$ we get

$$\frac{d^2\bar{\phi}_2}{dz^2} + \left[\frac{\omega^2}{\beta_2^2} - k^2 + \frac{1}{4\mu_2^2}\left(\frac{d\mu_2}{dz}\right)^2 - \frac{1}{2\mu_2}\,\frac{d^2\mu_2}{dz^2}\right]\bar{\phi}_2 = 0 \tag{4}$$

Restoring the values of $\beta_2^2$, $\mu_2$ in (4) and substituting $\zeta = \tanh(pz - \lambda)$, the equation (4) reduces to

$$\frac{d^2\bar{\phi}_2}{d\zeta^2} - \frac{2\zeta}{1-\zeta^2}\,\frac{d\bar{\phi}_2}{d\zeta} + \frac{\nu(\nu+1)-\gamma^2-\nu(\nu+1)\zeta^2}{(1-\zeta^2)^2}\,\bar{\phi}_2 = 0 \tag{5}$$

where,
$$\left.\begin{aligned} \nu &= -\tfrac{1}{2} \pm \frac{\omega}{p}\sqrt{(\rho/\mu)} \\ \text{and}\quad \gamma &= \pm\sqrt{(p^2+4k^2)}/2p \end{aligned}\right\} \tag{6}$$

Equation (5) is the well known Legendre's associated equation and the solution is given by—

$$\bar{\phi}_2 = A_2 P_\nu^\gamma(\zeta) + B_2 Q_\nu^\gamma(\zeta) \tag{7}$$

where,

$$P_\nu^\gamma(\zeta) = 2^\gamma\sqrt{\pi}\,(\zeta^2-1)^{-\frac{1}{2}\gamma}\left\{\frac{F\left(-\frac{\nu}{2}-\frac{\gamma}{2},\,\frac{1}{2}+\frac{\nu}{2}-\frac{\gamma}{2};\,\frac{1}{2};\,\zeta^2\right)}{\Gamma\left(\frac{1}{2}-\frac{\nu}{2}-\frac{\gamma}{2}\right)\Gamma\left(1+\frac{\nu}{2}-\frac{\gamma}{2}\right)}\right. -$$

$$\left. -2\zeta\,\frac{F\left(\frac{1}{2}-\frac{\nu}{2}-\frac{\gamma}{2},\,1+\frac{\nu}{2}-\frac{\gamma}{2};\,\frac{3}{2};\,\zeta^2\right)}{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}-\frac{\gamma}{2}\right)\Gamma\left(-\frac{\nu}{2}-\frac{\gamma}{2}\right)}\right\} \qquad |\zeta^2|<1$$

and

$$Q_\nu^\gamma(\zeta) = \sqrt{\pi}\,e^{i\gamma\pi}\,2^\gamma(\zeta^2-1)^{-\frac{1}{2}\gamma}\left\{\frac{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}+\frac{\gamma}{2}\right)}{2\,\Gamma\left(1+\frac{\nu}{2}-\frac{\gamma}{2}\right)}e^{\pm i\frac{1}{2}\pi(\gamma-\nu-1)}\,F\left(-\frac{\nu}{2}-\frac{\gamma}{2},\frac{1}{2}+\frac{\nu}{2}-\frac{\gamma}{2};\frac{1}{2};\zeta^2\right)+\right.$$

$$\left. +\frac{\zeta\,\Gamma\left(1+\frac{\nu}{2}+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}-\frac{\gamma}{2}\right)}\,e^{\pm i\frac{1}{2}\pi(\gamma-\nu)}\,F\left(1-\frac{\nu}{2}-\frac{\gamma}{2},\,1+\frac{\nu}{2}-\frac{\gamma}{2};\,\frac{3}{2};\,\zeta^2\right)\right\}$$

$$|\zeta^2| < 1$$

where, $F_1(\alpha, \beta; \gamma; z)$ is the hypergeometric function. Since $|\zeta^2| = |\tanh^2(pz-\lambda)| < 1$, the solution is true for all values of $z$. The functions $P_\nu^{i\gamma}(\zeta)$ and $Q_\nu^{i\gamma}(\zeta)$ are known as Legendre's functions of the first and second kind respectively.

Hence,
$$\phi_2 = \frac{(1-\zeta^2)^{1/4}}{\sqrt{\mu}} \left[ A_2 P_\nu^{i\gamma}(\zeta) + B_2 Q_\nu^{i\gamma}(\zeta) \right] \tag{9}$$

The equations of motion in the first layer and the half-space are—

$$\rho_j \frac{\partial^2 v_j}{\partial t^2} = \mu_j \nabla^2 v_j \qquad j = 1, 3$$

Let us assume $v_j = \phi_j(z)\cos(\omega t - kx)$, then $\phi_j$ is a solution of the equation—

$$\frac{d^2\phi_j}{dz^2} + \left[\frac{\omega^2}{\beta_j^2} - k^2\right]\phi_j = 0 \tag{10}$$

Therefore, the solutions in the first layer and the half-space are given by—

$$\left.\begin{aligned} \phi_1 &= A_1 \cos(k s_1 z) + B_1 \sin(k s_1 z) \\ \text{and} \quad \phi_3 &= A_3 e^{iks_3 z} + B_3 e^{-iks_3 z} \end{aligned}\right\} \tag{11}$$

where, $s_i = \left\{\left( c^2/\beta_i^2 \right) - 1\right\}^{\frac{1}{2}} \qquad i = 1, 3$

Assuming $s_3$ to be positive imaginary, and since $\phi_3 \to 0$ as $z \to \infty$, $B_3 = 0$

$$\text{hence,} \quad \phi_3 = A_3 e^{iks_3 z} \tag{12}$$

The boundary conditions are—

$$\tau_{zy}^{[1]}\Big|_{z=-H_1} = M(k)\cos(\omega t - kx) \tag{13(a)}$$

$$\phi_1\Big|_{z=0} = \phi_2\Big|_{z=0} \tag{13(b)}$$

$$\tau_{zy}^{[1]}\Big|_{z=0} = \tau_{zy}^{[2]}\Big|_{z=0} \tag{13(c)}$$

$$\phi_2\Big|_{z=H_2} = \phi_3\Big|_{z=H_2} \tag{13(d)}$$

and
$$\tau_{zy}^{[2]}\Big|_{z=H_2} = \tau_{zy}^{[3]}\Big|_{z=H_2} \tag{13(e)}$$

where,
$$\tau_{zy}^{[j]} = \mu_j \frac{\partial \phi_j}{\partial z}\cos(\omega t - kx) \quad ; \quad j = 1, 2, 3$$

From the above boundary conditions given in 13(a) to 13(e), $A_i$ , $B_j$ $(i = 1, 2, 3;\ j = 1, 2)$

can be written in the following way:

$$\begin{bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} ks_1 \sin(ks_1 H_1) & ks_1 \cos(ks_1 H_1) & 0 & 0 & 0 \\ 1 & 0 & -\sqrt{\frac{\operatorname{sech}\lambda}{\mu}}\, P_\nu^\gamma \times (-\tanh\lambda) & -\sqrt{\frac{\operatorname{sech}\lambda}{\mu}}\, Q_\nu^\gamma \times (-\tanh\lambda) & 0 \\ 0 & ks_1\mu_1 & -p\sqrt{\mu}\, f_1(-\lambda) \times \operatorname{sech}\lambda & -p\sqrt{\mu}\, f_2(-\lambda) \times \operatorname{sech}\lambda & 0 \\ 0 & 0 & -\sqrt{\frac{\operatorname{sech}(pH_2-\lambda)}{\mu}} \times P_\nu^\gamma \tanh(pH_2-\lambda) & -\sqrt{\frac{\operatorname{sech}(pH_2-\lambda)}{\mu}} \times Q_\nu^\gamma \tanh(pH_2-\lambda) & e^{iks_3H_2} \\ 0 & 0 & -p\sqrt{\mu}\, f_1(pH_2-\lambda) \times \operatorname{sech}(pH_2-\lambda) & -p\sqrt{\mu}\, f_2(pH_2-\lambda) \times \operatorname{sech}(pH_2-\lambda) & \mu_3 iks_3 e^{iks_3H_2} \end{bmatrix}^{-1} \times$$

$$\times \begin{bmatrix} M(k) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \psi_1(k) \\ \chi_1(k) \\ \psi_2(k) \\ \chi_2(k) \\ \psi_3(k) \end{bmatrix} \text{ say} \tag{14}$$

where,

$$f_1(-\lambda) = \frac{d}{d\zeta}\left[ (1-\zeta^2)^{1/4} P_{\nu}^{\gamma} \; (\zeta) \right]_{\zeta = -\tanh\lambda}$$

$$= \operatorname{sech}^{-3/2}\lambda \left[ (\nu+\gamma) \, P_{\nu-1}^{\gamma} \; (-\tanh\lambda) + (\tfrac{1}{2}+\nu) \tanh\lambda \, P_{\nu}^{\gamma} \; (-\tanh\lambda) \right]$$

and

$$f_2(-\lambda) = \frac{d}{d\zeta}\left[ (1-\zeta^2)^{1/4} Q_{\nu}^{\gamma} \; (\zeta) \right]_{\zeta = -\tanh\lambda}$$

$$= \operatorname{sech}^{-3/2}\lambda \left[ (\nu+\gamma) \, Q_{\nu-1}^{\gamma} (-\tanh\lambda) + (\tfrac{1}{2}+\nu) \tanh\lambda \, Q_{\nu}^{\gamma} (-\tanh\lambda) \right] \quad (15)$$

To have a concentrated line source on $x = 0$, let us assume

$$M(k) = \frac{Ldk}{\pi}$$

and integrate with respect to $k$ from $-\infty$ to $\infty$

*i.e.*,
$$\tau_{zy} \Big|_{Z=-H_1} = \frac{L}{\pi} \int_{-\infty}^{\infty} \cos(\omega t - kx)\, dk = L\delta(x)\cos(\omega t)$$

where $\delta(x)$ is the Dirac's delta function.

The displacements are, therefore, given by —

$$\left.\begin{aligned} v_1 &= [\, \overline{A_1} \cos(ks_1 z) + \overline{B_1} \sin(ks_1 z) \,] \cos \omega t \\ v_2 &= \sqrt{\frac{\operatorname{sech}(pz-\lambda)}{\bar{\mu}}} \left[ \overline{A}_2 \;\; P_{\nu}^{\gamma} \; (\tanh \overline{ph-\lambda}) + B_2 Q_{\nu}^{\gamma} \; (\tanh \overline{pz-\lambda}) \right] \cos \omega t \\ \text{and} \quad v_3 &= \overline{A}_3 e^{iks_3 z} \cos \omega t \end{aligned}\right\} \quad (16)$$

where,

$\overline{A_i}$, $\overline{B_j}$ $\quad (i = 1, 2, 3;\ j = 1, 2)$ are —

$$\left.\begin{aligned} \overline{A_i} &= \frac{1}{\pi} \int_{-\infty}^{\infty} \psi_i(k)\, dk \\ \overline{B_j} &= \frac{1}{\pi} \int_{-\infty}^{\infty} \chi_j(k)\, dk \end{aligned}\right\} \quad (17)$$

and

$$\triangle(k)=\begin{vmatrix} ks_1\sin(ks_1H_1) & ks_1\cos(ks_1H_1) & 0 & 0 & 0 \\ 1 & 0 & \sqrt{\frac{\operatorname{sech}\lambda}{\bar{\mu}}}\,P_\nu^{\gamma}\times(-\tanh\lambda) & \sqrt{\frac{\operatorname{sech}\lambda}{\bar{\mu}}}\,Q_\nu^{\gamma}\times(-\tanh\lambda) & 0 \\ 0 & ks_1\mu_1 & p\sqrt{\bar{\mu}}\,f_1\times(-\lambda)\operatorname{sech}\lambda & p\sqrt{\bar{\mu}}\,f_2\times(-\lambda)\operatorname{sech}\lambda & 0 \\ 0 & 0 & \sqrt{\frac{\operatorname{sech}(pH_2-\lambda)}{\bar{\mu}}}\times P_\nu^{\gamma}\tanh(pH_2-\lambda) & \sqrt{\frac{\operatorname{sech}(pH_2-\lambda)}{\bar{\mu}}}\times Q_\nu^{\gamma}(\tanh pH_2-\lambda) & 1 \\ 0 & 0 & p\sqrt{\bar{\mu}}\,f_1(pH_2-\lambda)\times\operatorname{sech}(pH_2-\lambda) & p\sqrt{\bar{\mu}}\,f(pH_2-\lambda)\times\operatorname{sech}(pH_2-\lambda) & \mu_3iks_3 \end{vmatrix}=0 \quad (18)$$

gives the frequency equation of the model under study.

## 2. Dicussion

The phase velocity curve for different frequencies can be obtained by solving the equation $\triangle(k)=0$. Since $\omega=kc$, where $c$ is the phase velocity of the Love wave, a frequency curve correlating $\omega$ and $c$ can be drawn and the properties of the heterogeneous layer can be studied.

The frequency equation on simplification may be written in the following form—

$$k\sqrt{\frac{c^2}{\beta_1^2}-1}\;\tanh kH_1\sqrt{\frac{c^2}{\beta_1^2}-1}$$

$$=\frac{p\bar{\mu}}{\mu_1}(\operatorname{sech}\lambda)^{\frac{1}{2}}\times\frac{f_1(-\lambda)X_1-f_2(-\lambda)X_2}{P_\nu^{\gamma}(-\tanh\lambda)X_1-Q_\nu^{\gamma}(-\tanh\lambda)X_2} \quad (19)$$

where

$$X_1=\begin{vmatrix}\sqrt{\frac{\operatorname{sech}(pH_2-\lambda)}{\bar{\mu}}}\,Q_\nu^{\gamma}(\tanh\overline{pH_2-\lambda}) & 1 \\ p(\bar{\mu})^{\frac{1}{2}}\,f_1(pH_2-\lambda)\operatorname{sech}(pH_2-\lambda) & \mu_3iks_3\end{vmatrix}$$

$$X_2=\begin{vmatrix}\sqrt{\frac{\operatorname{sech}(pH_2-\lambda)}{\bar{\mu}}}\,P_\nu^{\gamma}(\tanh\overline{pH_2-\lambda}) & 1 \\ p(\bar{\mu})^{\frac{1}{2}}\,f_1(pH_2-\lambda)\operatorname{sech}(pH_2-\lambda) & \mu_3iks_3\end{vmatrix}$$

Equation (19) gives an implicit relation between the wave number $k$ and the phase velocity $c$ of the Love wave and hence $c$ may be obtained as a function of the period $T$ through the relation $T = 2\pi/kc$.

When $c \rightarrow \beta_1$, from equation (19) it follows

$$kH_1 \longrightarrow (n + \tfrac{1}{2})\pi \Big/ \sqrt{\frac{c^2}{\beta_1^2} - 1} \longrightarrow \infty$$

and it is seen that propagation at the phase velocity $c = \beta_1$ occurs in each mode at the highest frequencies. More precisely from the definition of the wave number $k = 2\pi/l$ large values of $kH_1$, correspond to the large values of the dimensionless parameter $H_1/l$. Thus for $l << H_1$, $c \rightarrow \beta_1$ for all modes.

Again when

$$c \longrightarrow \beta_2$$

$$\tan kH_1 \sqrt{\frac{\beta_2^2}{\beta_1^2} - 1}$$

$$= \frac{p\bar{\mu}}{\mu_1} \sqrt{\operatorname{sech}\lambda} \; \frac{1}{\sqrt{\frac{\beta_2^2}{\beta_1^2} - 1}} \times \underset{c \rightarrow \beta_2}{\mathrm{Lt}} \; \frac{f_1(-\lambda) X_1 - f_2(-\lambda) X_2}{P_\nu^{\prime}(-\tanh\lambda) X_1 - Q_\nu^{\prime}(-\tanh\lambda) X_2}$$

$$= \tan \chi \text{ (say)} \qquad \left(\frac{\pi}{2} \leqslant |\chi| \leqslant 0\right)$$

Hence,

$$k_n H_1 \longrightarrow \frac{n\pi + \chi}{\sqrt{\frac{\beta_2^2}{\beta_1^2} - 1}}$$

The range of $k_n H_1$ is such that

$$n\pi < k_n H_1 \sqrt{\frac{c^2}{\beta_1^2} - 1} < n(n + \tfrac{1}{2})\pi$$

Thus in all probability it is concluded that as $n \rightarrow \infty$, that is, for higher modes

$$k_n H_1 \longrightarrow n\pi \Big/ \sqrt{\frac{c^2}{\beta_1^2} - 1}$$

and the modes form a harmonic series.

It may also be seen that when the heterogeneous layer is absent, *i.e.*, when $H_2 = 0$, the problem reduces to a well known problem of Love wave propagation in a homogeneous layer over a half-space and the frequency equation in this case from Eq. (18) reduces to

$$\tan kH_1 \sqrt{\frac{c^2}{\beta_1^2} - 1} = \frac{\mu_3}{\mu_1} \left[ (1 - c^2/\beta_3^2)^{\frac{1}{2}} / (c^2/\beta_1^2 - 1)^{\frac{1}{2}} \right]$$

**4. Acknowledgement**

The author is thankful to Dr. A.N. Tandon for his guidance and encouragement. Thanks are also due to Shri H. M. Chaudhury and Shri S.N. Bhattacharya for helpful suggestions and discussions.

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