

On the propagation of spherical waves in a viscoelastic medium

S. N. BHATTACHARYA and S. N. CHATTERJEE

Meteorological Office, New Delhi

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ABSTRACT. Theory of calculating the displacement in a general viscoelastic medium, generated by an arbitrary pressure applied on the surface of a spherical cavity have been developed. In particular, three types of pressures have been considered and the corresponding displacements in Maxwell as well as Voigt type viscoelastic media have been derived. As special cases, the results of earlier authors in elastic medium have been deduced and compared. The results obtained have been discussed.

1. Introduction

Spherical wave propagation in an elastic medium was studied by various authors (Jeffreys 1931, Sharpe 1942 and Blanke 1952). It is quite well known that earth is not a perfect elastic body. A wide variety of earth materials such as silts, clays, sands and shales do not show perfect elastic behaviour under any kind of loading. The non-elastic behaviour of the earth results in attenuation of seismic signals with distance and attenuation of free oscillations with time. Although the effect of non-elastic behaviour is relatively small for earthquake waves, but quite pronounced for explosion generated high frequency waves (Ewing, Jardetzky and Press 1957).

In this paper we have studied the problem of wave propagation when different types of pressures were applied on the surface of a spherical cavity in a general viscoelastic medium. Mattice and Lieber (1954) and Chakrabarty (1961) considered this problem in Voigt type viscoelastic medium. Horton (1959) argued that Voigt solid is not a satisfactory representation of the elastic properties of the earth and he proposed a modification of Voigt solid. Since no satisfactory model representing the viscoelastic behaviour of earth for all frequencies is available, it is felt necessary to study the problem in a general viscoelastic medium. Detailed analysis of the problem in Maxwell as well as Voigt media has been presented. As special cases the results of Blanke (1952) and Jeffreys (1931) in elastic medium have been deduced.

2. Equation of motion and boundary condition

The stress-strain relations in a general viscoelastic medium are given by—

$$P S_{kl}(t) = 2Q e_{kl}(t) \quad (1)$$

in case of pure shear stress and

$$P' \sigma(t) = 3Q' \epsilon(t) \quad (2)$$

in case of pressure-volume changes, where P , Q , P' and Q' are operators given by—

$$P = \sum_{j=0}^m p_j \frac{\partial^j}{\partial t^j}, \quad Q = \sum_{j=0}^n q_j \frac{\partial^j}{\partial t^j}$$

$$P' = \sum_{j=0}^m p'_j \frac{\partial^j}{\partial t^j}, \quad Q' = \sum_{j=0}^n q'_j \frac{\partial^j}{\partial t^j}$$

$p_0 = p'_0 = 1$, Stress $\sigma_{kl}(t)$ and strain $\epsilon_{kl}(t)$ are

$$\begin{aligned} \sigma_{kl} &= S_{kl}(t) + \delta_{kl} \sigma(t), \\ \epsilon_{kl} &= e_{kl}(t) + \delta_{kl} \epsilon(t) \end{aligned} \quad (3)$$

where,

$$\begin{aligned}\epsilon_{kl} &= \frac{1}{2} \left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right), \\ \sigma &= \frac{1}{3} (\sigma_{11} + \sigma_{22} + \sigma_{33}), \\ \epsilon &= \frac{1}{3} (\epsilon_{11} + \epsilon_{22} + \epsilon_{33}),\end{aligned}$$

\vec{u} (u_1, u_2, u_3) being the displacement vector.

Using (1), (2) and (3) the equation of motion is given by (Newman 1951, Alfrey and Gurnee 1956)

$$PP' \left(\rho \frac{\partial^2 \vec{u}}{\partial t^2} \right) = \nabla \cdot \nabla \left(\frac{4}{3} P'Q + PQ' \right) \vec{u} - \nabla \times \nabla \times P'Q \vec{u} \quad (4)$$

In the further analysis spherical coordinates (r, θ, ϕ) with origin at the centre of cavity will be used. The boundary condition is —

$$-\sigma_{rr}|_{r=a} = P(t) \quad (5)$$

' a ' being the radius of the cavity. Assuming the medium to be homogeneous, isotropic and infinite, the boundary condition shows that the motion is spherically symmetrical. The radial displacement u_r may be supposed a —

$$u_r = \frac{\partial \phi}{\partial r}$$

The equation of motion (4) then becomes —

$$PP' \frac{\partial^2 (r\phi)}{\partial t^2} = \frac{1}{\rho} \left(\frac{4}{3} P'Q + PQ' \right) \frac{\partial^2 (r\phi)}{\partial r^2} \quad (6)$$

3. Solution

Taking

$$\phi = \frac{1}{r} f(r) \exp(-i\omega t)$$

the equation (6) becomes —

$$\frac{\partial^2 f(r)}{\partial r^2} = \frac{\omega^2 \sum_0^m p_j (-i\omega)^j \sum_0^m p_j' (-i\omega)^j}{\frac{1}{\rho} \left[\frac{4}{3} \sum_0^m p_j' (-i\omega)^j \sum_0^n q_j (-i\omega)^j + \sum_0^m p_j (-i\omega)^j \sum_0^n q_j' (-i\omega)^j \right]} f(r) \quad (7)$$

so that,

$$f(r) = \exp[\pm i\omega X r]$$

where,

$$X = \left\{ \frac{\sum_0^m p_j (-i\omega)^j \sum_0^m p_j' (-i\omega)^j}{\frac{1}{\rho} \left[\frac{4}{3} \sum_0^m p_j' (-i\omega)^j \sum_0^n q_j (-i\omega)^j + \sum_0^m p_j (-i\omega)^j \sum_0^n q_j' (-i\omega)^j \right]} \right\}^{\frac{1}{2}}$$

$$= \left[\frac{\rho}{\frac{4}{3} \sum_0^n q_j(-i\omega)^j \left| \sum_0^m p_j(-i\omega)^j + \sum_0^n q_j'(-i\omega)^j \right| \sum_0^m p_j'(-i\omega)^j} \right]^{\frac{1}{2}}$$

Considering u_r vanishes when r is infinite, the solution of (6) is —

$$\phi = \frac{1}{r} \exp[-\gamma(r-a)] \exp[-i\omega\{t-\beta(r-a)\}] \tag{7a}$$

where,

$$\gamma = |\omega \operatorname{Im} X|,$$

$$\beta = \operatorname{Re} X$$

The sum of the right hand side of (7) multiplied by functions independent of r and t also remains solution of (6). Thus —

$$\phi = \frac{1}{2\pi r} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(\omega) P(\xi) \exp[-\gamma(r-a)] \exp[-i\omega\{t-\beta(r-a)-\xi\}] d\omega d\xi \tag{8}$$

The boundary condition (5) can be written as —

$$\begin{aligned} -\sigma_{rr} \Big|_{r=a} &= P(t) \\ &= \int_{-\infty}^{\infty} \bar{P}(\omega) \exp(-i\omega t) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(\xi) \exp[-i\omega(t-\xi)] d\omega d\xi \end{aligned} \tag{9}$$

where,

$$\bar{P}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} P(\xi) \exp(i\omega\xi) d\xi$$

Using (1) and (2) we note that —

$$PP'\sigma_{rr} = \frac{4}{3} P'Q \left(\frac{du_r}{dr} - \frac{u_r}{r} \right) + PQ' \left(\frac{du_r}{dr} + \frac{2u_r}{r} \right) \tag{10}$$

Hence using (10) and (8), the boundary condition (9) gives —

$$\begin{aligned} &\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(\omega) \left[\frac{4}{3} \sum_0^m p_j'(-i\omega)^j \sum_0^n q_j(-i\omega)^j \left\{ \frac{3}{a^3} + \frac{3\gamma}{a^2} - \frac{3i\omega\beta}{a^2} + \frac{\gamma^2}{a} - \frac{2i\omega\beta\gamma}{a} - \frac{\omega^2\beta^2}{a} \right\} \right. \\ &\quad \left. + \sum_0^m p_j(-i\omega)^j \sum_0^n q_j'(-i\omega)^j \left\{ \frac{\gamma^2}{a} - \frac{2i\omega\beta\gamma}{a} - \frac{\omega^2\beta^2}{a} \right\} \right] P(\xi) \exp\{-i\omega(t-\xi)\} d\omega d\xi \\ &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\sum_0^m p_j(-i\omega)^j \sum_0^m p_j'(-i\omega)^j \right] P(\xi) \exp\{-i\omega(t-\xi)\} d\omega d\xi \end{aligned}$$

which gives—

$$A(\omega) = \frac{-1}{4 \frac{\sum_0^n q_j (-i\omega)^j}{\sum_0^m p_j (-i\omega)^j} \left\{ \frac{1}{a^3} + \frac{\gamma}{a^2} - \frac{i\omega\beta}{a^2} \right\} + \left\{ \frac{4}{3} \frac{\sum_0^n q_j (-i\omega)^j}{\sum_0^m p_j (-i\omega)^j} + \frac{\sum_0^n q_j' (-i\omega)^j}{\sum_0^m p_j' (-i\omega)^j} \right\} \left[\frac{\gamma^2}{a} - \frac{2i\omega\beta\gamma}{a} - \frac{\omega^2\beta^2}{a} \right]} \quad (11)$$

Substituting $A(\omega)$ from (11) in (8) the expression for the displacement potential ϕ for a general viscoelastic medium can be obtained.

We shall consider three types pressures at the surface of the cavity $r = a$.

Case I

$$\begin{aligned} P(t) &= P_0 \exp(-\lambda_0 t) & , & \quad t \geq 0 \\ &= 0 & , & \quad t < 0 \end{aligned} \quad (12)$$

Then (8) gives—

$$\begin{aligned} \phi &= \frac{P_0}{2\pi r} \int_{-\infty}^{\infty} A(\omega) \exp[-\gamma(r-a)] \exp[-i\omega\{t-\beta(r-a)\}] d\omega \int_0^{\infty} \exp\{i(\omega+i\lambda_0)\xi\} d\xi \\ &= -\frac{iP_0}{2\pi r} \int_{-\infty}^{\infty} \frac{A(\omega)}{\omega+i\lambda_0} \exp[-\gamma(r-a)] \exp[-i\omega\{t-\beta(r-a)\}] d\omega \end{aligned} \quad (13)$$

Case II

$$\begin{aligned} P(t) &= P_0 t \exp(-\lambda_0 t) & , & \quad t \geq 0 \\ &= 0 & , & \quad t < 0 \end{aligned} \quad (14)$$

Then,

$$\begin{aligned} \phi &= \frac{P_0}{2\pi r} \int_{-\infty}^{\infty} A(\omega) \exp[-\gamma(r-a)] \exp[-i\omega\{t-\beta(r-a)\}] d\omega \int_0^{\infty} \xi \exp\{i(\omega+i\lambda_0)\xi\} d\xi \\ &= -\frac{P_0}{2\pi r} \int_{-\infty}^{\infty} \frac{A(\omega)}{(\omega+i\lambda_0)^2} \exp[-\gamma(r-a)] \exp[-i\omega\{t-\beta(r-a)\}] d\omega \end{aligned} \quad (15)$$

Case III

$$P(t) = P_0 \delta(t) \quad (16)$$

Then,

$$\phi = \frac{P_0}{2\pi r} \int_{-\infty}^{\infty} A(\omega) \exp[-\gamma(r-a)] \exp[-i\omega\{t-\beta(r-a)\}] d\omega \quad (17)$$

4. Particular models

In this section we shall consider displacement potential and displacement in Maxwell and Voigt media.

4.1. Maxwell medium

Here the operators P , Q , P' and Q' are—

$$P = \frac{G}{\eta} + \frac{\partial}{\partial t} \quad , \quad Q = G \frac{\partial}{\partial t}$$

$$P' = \frac{G'}{\eta'} + \frac{\partial}{\partial t} \quad , \quad Q' = G' \frac{\partial}{\partial t}$$

Hence X is given by—

$$X^2 = \frac{\rho}{\frac{4}{3} \frac{-Gi\omega}{\eta - i\omega} + \frac{-G'i\omega}{\eta' - i\omega}}$$

$$= \frac{\rho}{\frac{4}{3} G \left(1 + i \frac{G}{\eta\omega}\right)^{-1} + G' \left(1 + i \frac{G'}{\eta'\omega}\right)^{-1}}$$

$$\approx \frac{\rho}{\left(\frac{4}{3} G + G'\right) - \frac{i}{\omega} \left(\frac{4}{3} \frac{G^2}{\eta} + \frac{G'^2}{\eta'}\right)} \quad , \text{ neglecting higher orders of } G/\eta \text{ and } G'/\eta'$$

$$= \frac{1}{c^2 - \frac{i}{\omega} d^2}$$

where, $c^2 = \left(\frac{4}{3} G + G'\right) / \rho$ and $d^2 = \left(\frac{4}{3} \frac{G^2}{\eta} + \frac{G'^2}{\eta'}\right) / \rho$. Hence,

$$X = \frac{1}{c} \left[1 - \frac{i}{\omega} \frac{d^2}{c^2} \right]^{-\frac{1}{2}} \approx \frac{1}{c} \left[1 + \frac{i}{\omega} \frac{d^2}{2c^2} \right]$$

Thus,

$$\beta = \frac{1}{c} \tag{18a}$$

and

$$\gamma = \frac{d^2}{2c^3} \tag{18b}$$

$A(\omega)$ is given by,

$$A(\omega) = \frac{-1}{4 \frac{-Gi\omega}{\eta - i\omega} \left\{ \frac{1}{a^3} + \frac{\gamma}{a^2} - \frac{i\omega\beta}{a^2} \right\} + \left\{ \frac{4}{3} \frac{-Gi\omega}{\eta - i\omega} + \frac{-G'i\omega}{\eta' - i\omega} \right\} \left\{ \frac{\gamma^2}{a} - \frac{2i\omega\beta\gamma}{a} - \frac{\omega^2\beta^2}{a} \right\}}$$

$$= \frac{-i\omega}{4G \left(i\omega + \frac{G}{\eta}\right) \left(\frac{1}{a^3} + \frac{\gamma}{a^2} - \frac{i\omega\beta}{a^2}\right) + \rho(c^2 i\omega + d^2) \left(\frac{\gamma^2}{a} - \frac{2i\omega\beta\gamma}{a} - \frac{\omega^2\beta^2}{a}\right)}$$

$$= \frac{a}{\rho} \frac{1}{\omega^2 + \left(\frac{4G}{ac\rho} + \frac{2d^2}{c^2}\right) i\omega - \frac{4G}{\rho} \left(\frac{1}{a^2} + \frac{d^2}{2c^3 a}\right)}$$

where higher orders of G/η , G'/η' , $G/a\sqrt{\eta}$ have been neglected.

Hence,

$$A(\omega) = \frac{a}{\rho} \frac{1}{[\omega - (-iA_1 - B_1)] [\omega - (-iA_1 + B_1)]} \quad (19)$$

where,

$$A_1 = \frac{2G}{ac\rho} + \frac{d^2}{c^2}$$

$$B_1 = \left[\frac{4G}{\rho} \left(\frac{1}{a^2} + \frac{d^2}{2c^3 a} \right) - \left(\frac{2G}{ac\rho} + \frac{d^2}{c^2} \right)^2 \right]^{\frac{1}{2}}$$

We shall assume B_1 to be real positive quantity so that both the singularities of $A(\omega)$ lie in the lower half of the real axis in the complex ω -plane. In the case when B_1 is imaginary we have $Im A_1 > Im B_1$ and two singularities still lie in the lower half, but on the imaginary axis. Hence the following results can be easily evaluated for the latter case by suppressing B_1 and giving a new value to A_1 .

Case I

Substituting (19) in (13) the displacement potential ϕ for the pressure prescribed in (12) at the surface of the cavity is given by —

$$\phi = - \frac{i P_0 a \exp \left[- \frac{d^2}{2c^3} (r-a) \right]}{2\pi\rho r} \int_{-\infty}^{\infty} \frac{\exp [-i\omega\tau] d\omega}{(\omega + i\lambda_0) [\omega - (-iA_1 - B_1)] [\omega - (-iA_1 + B_1)]} \quad (20)$$

Where, $\tau = t - (r-a)/c$. When $\tau > 0$, performing integration of (18) in the complex ω -plane along a contour C consisting of,

(i) real axis from $\omega = +R$ to $\omega = -R$

(ii) lower half of the semi-circle $|\omega| = R$ as shown in Fig. 1(p. 562), we get —

$$\oint_C \frac{\exp [-i\omega\tau] d\omega}{(\omega + i\lambda_0) [\omega - (-iA_1 - B_1)] [\omega - (-iA_1 + B_1)]}$$

$$= \int_{-R}^R \frac{\exp [-i\omega\tau] d\omega}{(\omega + i\lambda_0) [\omega - (-iA_1 - B_1)] [\omega - (-iA_1 + B_1)]} +$$

$$+ \int_0^\pi \frac{\exp [-i\tau R \exp(i\theta)] iR \exp(i\theta) d\theta}{[R \exp(i\theta) + i\lambda_0] [R \exp(i\theta) - (-iA_1 - B_1)] [R \exp(i\theta) - (-iA_1 + B_1)]}$$

$$= I_1 + I_2, \quad (\text{say})$$

$$= 2\pi i \Sigma \text{ residues.}$$

Since, as $R \rightarrow \infty$, $I_2 \rightarrow 0$, we get from (20)

$$\phi = \frac{P_0 a \exp \left[- \frac{d^2}{2c^3} (r-a) \right]}{\rho r} \left[\frac{\exp(-\lambda_0 \tau)}{\{-i\lambda_0 + (iA_1 + B_1)\} \{-i\lambda_0 - (-iA_1 + B_1)\}} - \frac{\exp\{i(iA_1 + B_1)\tau\}}{2B_1(-iA_1 - B_1 + i\lambda_0)} + \frac{\exp\{i(iA_1 - B_1)\tau\}}{2B_1(-iA_1 + B_1 + i\lambda_0)} \right]$$

$$\begin{aligned}
 &= \frac{P_0 a \exp \left[-\frac{d^2}{2c^3} (r-a) \right]}{\rho r} \left[-\frac{\exp(-\lambda_0 \tau)}{(A_1 - \lambda)^2 + B_1^2} + \frac{\exp \{i(iA_1 + B_1)\tau\}}{2B_1 \{i(A_1 - \lambda_0) + B_1\}} - \right. \\
 &\quad \left. - \frac{\exp \{i(iA_1 - B_1)\tau\}}{2B_1 \{i(A_1 - \lambda_0) - B_1\}} \right] \\
 &= \frac{P_0 a \exp \left[-\frac{d^2}{2c^3} (r-a) \right]}{\rho r [(A_1 - \lambda_0)^2 + B_1^2]} \left[-\exp(-\lambda_0 \tau) + \right. \\
 &\quad \left. + \frac{\sqrt{\{(A_1 - \lambda_0)^2 + B_1^2\}}}{B_1} \exp(-A_1 \tau) \sin \left(B_1 \tau + \tan^{-1} \frac{B_1}{A_1 - \lambda_0} \right) \right] \quad (21)
 \end{aligned}$$

When $\tau < 0$, on integrating in the complex ω -plane along a semi-circular contour on the upper half of real axis we get—

$$\phi = 0$$

Thus for $\tau > 0$ the displacement u_r is given by—

$$\begin{aligned}
 u_r &= \frac{\partial \phi}{\partial r} \\
 &= \frac{P_0 a \exp \left[-\frac{d^2}{2c^3} (r-a) \right]}{\rho [(A_1 - \lambda_0)^2 + B_1^2]} \left[\frac{1}{r^2} \exp(-\lambda_0 \tau) - \frac{1}{r^2} \frac{\sqrt{\{(A_1 - \lambda_0)^2 + B_1^2\}}}{B_1} \exp(-A_1 \tau) \times \right. \\
 &\quad \times \sin \left(B_1 \tau + \tan^{-1} \frac{B_1}{A_1 - \lambda_0} \right) + \frac{1}{rc} \left(\frac{d^2}{2c^2} - \lambda_0 \right) \exp(-\lambda_0 \tau) + \\
 &\quad + \frac{\sqrt{\{(A_1 - \lambda_0)^2 + B_1^2\}}}{B_1 cr} \sqrt{\left\{ \left(A_1 - \frac{d^2}{2c^2} \right)^2 + B_1^2 \right\}} \exp(-A_1 \tau) \times \\
 &\quad \left. \times \sin \left(B_1 \tau + \tan^{-1} \frac{B_1}{A_1 - \lambda_0} - \tan^{-1} \frac{B_1}{A_1 - \frac{d^2}{2c^2}} \right) \right] \quad (22)
 \end{aligned}$$

Elastic case

In elastic case $\eta = \eta' = \infty$, modulus of rigidity, $G = \mu$, and bulk modulus $G' = k$, so that $d^2 = 0$ and displacement potential ϕ in (21) becomes,

$$\begin{aligned}
 \phi &= \frac{P_0 a}{\rho r [(A_1 - \lambda_0)^2 + B_1^2]} \left[-\exp(-\lambda_0 \tau) + \exp(-A_1 \tau) \left\{ 1 + \left(\frac{A_1 - \lambda_0}{B_1} \right)^2 \right\}^{\frac{1}{2}} \times \right. \\
 &\quad \left. \times \cos \left(B_1 \tau - \tan^{-1} \frac{A_1 - \lambda_0}{B_1} \right) \right] \quad (23)
 \end{aligned}$$

where angular frequency is—

$$\begin{aligned}
 B_1 &= \sqrt{\left(\frac{4\mu}{\rho a^2} - \frac{4\mu^2}{a^2 c^2 \rho^2} \right)} = \frac{2}{a} \sqrt{\frac{\mu}{\rho}} \sqrt{\left(1 - \frac{\mu}{\lambda + 2\mu} \right)} \\
 &= \frac{2}{a} \sqrt{\frac{\mu}{\rho}} \sqrt{\left(\frac{\lambda + \mu}{\lambda + 2\mu} \right)} = \frac{c}{a} \frac{(1 - 2\bar{\sigma})^{\frac{1}{2}}}{1 - \bar{\sigma}} \quad , \quad (24) \\
 c^2 &= \frac{1}{\rho} \left(\frac{4}{3} \mu + k \right) = \frac{1}{\rho} (\lambda + 2\mu),
 \end{aligned}$$

$\lambda = k - \frac{2}{3}\mu$ and $\bar{\sigma}$ is the Poisson's ratio given by $\bar{\sigma} = \lambda/2(\lambda + \mu)$ and damping is —

$$A_1 = \frac{2\mu}{ac\rho} = \frac{2\mu}{a\sqrt{[\rho(\lambda + 2\mu)]}} = \frac{c}{a} \frac{1 - 2\bar{\sigma}}{1 - \bar{\sigma}} \quad (25)$$

The expression (23) for the displacement potential ϕ in elastic case is exactly similar to that obtained by Blanke (1952).

The displacement potential ϕ and the radial displacement u_r for a step function of pressure at the surface of the cavity can be obtained by putting $\lambda_0 = 0$ in (21) and in (22). Thus —

$$\phi = \frac{P_0 a \exp \left[-\frac{d^2}{2c^3} (r-a) \right]}{\rho r (A_1^2 + B_1^2)} \left[-1 + \left\{ 1 + \left(\frac{A_1}{B_1} \right)^2 \right\}^{\frac{1}{2}} \exp(-A_1 \tau) \sin \left(B_1 \tau + \tan^{-1} \frac{B_1}{A_1} \right) \right] \quad (26)$$

and

$$\begin{aligned} u_r = & \frac{P_0 a \exp \left[-\frac{d^2}{2c^3} (r-a) \right]}{\rho (A_1^2 + B_1^2)} \left\{ \frac{1}{r^2} - \frac{1}{r^2 B_1} \sqrt{[A_1^2 + B_1^2]} \exp(-A_1 \tau) \sin \left(B_1 \tau + \tan^{-1} \frac{B_1}{A_1} \right) + \right. \\ & + \frac{1}{r} \frac{d^2}{2c^3} + \frac{\sqrt{[A_1^2 + B_1^2]}}{B_1 c r} \sqrt{\left[\left(A_1 - \frac{d^2}{2c^3} \right)^2 + B_1^2 \right]} \times \\ & \left. \times \exp(-A_1 \tau) \sin \left(B_1 \tau + \tan^{-1} \frac{B_1}{A_1} - \tan^{-1} \frac{B_1}{A_1 - \frac{d^2}{2c^3}} \right) \right\} \quad (27) \end{aligned}$$

In elastic case, (26) gives —

$$\phi = \frac{P_0 a^3}{4\mu r} \left[-1 + \exp(-A_1 \tau) \left\{ \cos B_1 \tau + \frac{A_1}{B_1} \sin B_1 \tau \right\} \right] \quad (28)$$

where, B_1 and A_1 are given by (24) and (25) and may be written also in the forms —

$$B_1 = \frac{2}{a} \frac{v_s}{v_p} \sqrt{v_p^2 - v_s^2}$$

$$A_1 = \frac{2v_s^2}{a v_p}$$

where, $v_s = \sqrt{[\mu/\rho]}$ and $v_p = c = \sqrt{[(\lambda + 2\mu)/\rho]}$

The displacement u_r in this case, from (27) after simplification reduces to —

$$\begin{aligned} u_r = & \frac{P_0 a^3}{4\rho v_s^2} \left[\frac{1}{r^2} - \frac{1}{r^2} \exp \left(-\frac{2v_s^2 \tau}{a v_p} \right) \left\{ \cos B_1 \tau + \frac{v_s}{\sqrt{v_p^2 - v_s^2}} \sin B_1 \tau \right\} + \right. \\ & \left. + \frac{2v_s}{ra \sqrt{v_p^2 - v_s^2}} \exp \left(-\frac{2v_s^2 \tau}{a v_p} \right) \sin \left(\frac{2v_s}{a v_p} \sqrt{v_p^2 - v_s^2} \tau \right) \right] \\ = & \frac{P_0 a^3}{4\rho v_s^2} \left[\frac{1}{r^2} - \frac{1}{r^2} \exp \left(-\frac{2v_s^2 \tau}{a v_p} \right) \left\{ \cos B_1 \tau + \frac{v_s}{\sqrt{v_p^2 - v_s^2}} \sin B_1 \tau \right\} + \right. \\ & + \frac{1}{r} \exp \left(-\frac{2v_s^2 \tau}{a v_p} \right) \left\{ \left(\frac{2v_s^2}{a v_p^2} \cos B_1 \tau + \frac{B_1}{v_p} \sin B_1 \tau \right) - \right. \\ & \left. \left. - \frac{v_s}{\sqrt{v_p^2 - v_s^2}} \left(\frac{2v_s^2}{a v_p^2} \sin B_1 \tau - \frac{B_1}{v_p} \cos B_1 \tau \right) \right\} \right] \quad (29) \end{aligned}$$

The expressions (28) and (29) give the expressions of the displacement potential ϕ and the displacement u_r deduced by Jeffreys (1931) for a step function pressure pulse in elastic medium.

Case II

For the pressure prescribed in (14) we have,

$$\phi = -\frac{P_0 a}{2\pi r \rho} \exp\left[-\frac{d^2}{2c^3}(r-a)\right] \int_{-\infty}^{\infty} \frac{\exp[-i\omega\tau] d\omega}{\omega(+i\lambda_0)^2[\omega-(-iA_1-B_1)][\omega-(-iA_1+B_1)]} \quad (30)$$

When $\tau > 0$, we shall integrate (30) in the complex ω -plane along a contour as in Fig. 1. Here we may note that the point $\omega = -i\lambda_0$ is a pole of 2nd order. Thus performing the integration we have—

$$\begin{aligned} \phi &= \frac{P_0 a \exp\left[-\frac{d^2}{2c^3}(r-a)\right]}{r \rho B_1 \{(A_1-\lambda_0)^2 + B_1^2\}^2} \left[\{\tau[(A_1-\lambda_0)^2 + B_1^2] - 2(A_1-\lambda_0)\} \exp(-\lambda_0\tau) - \right. \\ &\quad \left. - \{(-[A_1-\lambda_0]^2 + B_1^2) \sin B_1\tau - 2B_1(A_1-\lambda_0) \cos B_1\tau\} \exp(-A_1\tau) \right] \\ &= \frac{P_0 a \exp\left[-\frac{d^2}{2c^3}(r-a)\right]}{\rho r \{(A_1-\lambda_0)^2 + B_1^2\}} \left[\left\{ \tau - \frac{2(A_1-\lambda_0)}{(A_1-\lambda_0)^2 + B_1^2} \right\} \exp(-\lambda_0\tau) - \right. \\ &\quad \left. - \frac{1}{B_1} \sin\left(B_1\tau - \tan^{-1} \frac{2B_1[A_1-\lambda_0]}{-(A_1-\lambda_0)^2 + B_1^2}\right) \exp(-A_1\tau) \right] \quad (31) \end{aligned}$$

When $\tau < 0$, on integrating in the complex ω -plane along a semi-circular contour on the upper half of the real axis we get—

$$\phi = 0$$

Thus for $\tau > 0$, the displacement u_r is given by—

$$\begin{aligned} u_r &= \frac{P_0 a \exp\left[-\frac{d^2}{2c^3}(r-a)\right]}{\rho [(A_1-\lambda_0)^2 + B_1^2]} \left[-\frac{1}{r^2} \left\{ \tau - \frac{2(A_1-\lambda_0)}{(A_1-\lambda_0)^2 + B_1^2} \right\} \exp(-\lambda_0\tau) + \right. \\ &\quad + \frac{1}{r^2 B_1} \sin\left(B_1\tau - \tan^{-1} \frac{2B_1[A_1-\lambda_0]}{-(A_1-\lambda_0)^2 + B_1^2}\right) \exp(-A_1\tau) + \\ &\quad + \frac{1}{rc} \left(\lambda_0 - \frac{d^2}{2c^2}\right) \left\{ \tau - \frac{2(A_1-\lambda_0)}{(A_1-\lambda_0)^2 + B_1^2} \right\} \exp(-\lambda_0\tau) - \\ &\quad - \frac{1}{B_1 r c} \sqrt{\left\{ \left(A_1 - \frac{d^2}{2c^2}\right)^2 + B_1^2 \right\}} \exp(-A_1\tau) \times \\ &\quad \times \sin\left(B_1\tau - \tan^{-1} \frac{2B_1[A_1-\lambda_0]}{-(A_1-\lambda_0)^2 + B_1^2} - \tan^{-1} \frac{B_1}{A_1 - \frac{d^2}{2c^2}}\right) \quad (32) \end{aligned}$$

Case III

For a pressure prescribed in (16) we have—

$$\phi = \frac{P_0 a \exp\left[-\frac{d^2}{2c^3}(r-a)\right]}{2\pi \rho r} \int_{-\infty}^{\infty} \frac{\exp[-i\omega\tau] d\omega}{[\omega-(-iA_1-B_1)][\omega-(-iA_1+B_1)]}$$

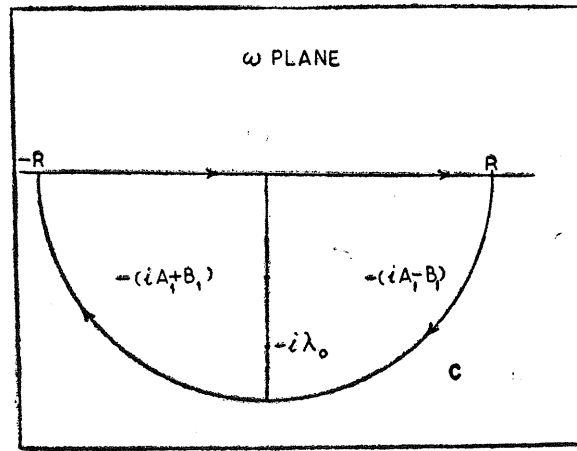


Fig. 1. Contour showing the path of integration

Choosing the contour as Fig. 1 with singularities at $\omega = -(iA_1 + B_1)$, $-iA_1 + B_1$ and performing integration we have —

$$\phi = \frac{P_0 a}{\rho r B_1} \exp \left[- \left\{ \tau A_1 + \frac{d^2}{2c^3} (r-a) \right\} \right] \sin (B_1 \tau), \quad \text{for } \tau > 0 \tag{33}$$

and as earlier,

$$\phi = 0, \quad \text{for } \tau < 0$$

Hence, for $\tau > 0$ the displacement u_r is given by —

$$\begin{aligned} u_r &= \frac{P_0 a}{\rho B_1} \exp \left[- \left\{ \tau A_1 + \frac{d^2}{2c^3} (r-a) \right\} \right] \times \\ &\quad \times \left[\frac{1}{r^2} \sin (B_1 \tau) - \frac{1}{rc} \left\{ \left(\frac{d^2}{2c^2} - A_1 \right) \sin (B_1 \tau) + \cos B_1 \tau \right\} \right] \\ &= \frac{P_0 a}{\rho B_1} \exp \left[- \left\{ \tau A_1 + \frac{d^2}{2c^3} (r-a) \right\} \right] \left[- \frac{1}{r^2} \sin (B_1 \tau) - \right. \\ &\quad \left. - \frac{1}{rc} \sqrt{\left\{ \left(\frac{d^2}{2c^2} - A_1 \right)^2 + B_1^2 \right\}} \sin \left(B_1 \tau + \tan^{-1} \frac{B_1}{\frac{d^2}{2c^2} - A_1} \right) \right] \end{aligned} \tag{34}$$

4.2. Voigt medium

Here the operators P, Q, P', Q' are given by —

$$P = 1, \quad Q = G + \eta \frac{\partial}{\partial t},$$

$$P' = 1, \quad Q' = G' + \eta' \frac{\partial}{\partial t}.$$

In this case,

$$X^2 = \frac{1}{\frac{1}{\rho} \left[\frac{4}{3} (G - i\omega\eta) + (G' - i\omega\eta') \right]} = \frac{1}{c^2 - i\omega h^2} \tag{35}$$

where, $c^2 = \frac{1}{\rho} \left(\frac{4}{3} G + G' \right)$ and $h^2 = \frac{1}{\rho} \left(\frac{4}{3} \eta + \eta' \right)$.

Neglecting higher orders of h^2/c^2 , (35) gives —

$$\beta = \frac{1}{c} \tag{36a}$$

and

$$\gamma = \frac{\omega^2 h^2}{2c^3} \tag{36b}$$

From (11) we have,

$$A(\omega) = \frac{-1}{4(G - i\omega\eta) \left(\frac{1}{a^3} + \frac{\gamma}{a^2} - \frac{i\omega\beta}{a^2} \right) + \rho(c^2 - ih^2\omega) \left(\frac{r^2}{a} - \frac{2i\omega\beta\gamma}{a} - \frac{\omega^2\beta^2}{a} \right)} \tag{37}$$

Substituting the values of β and γ from (36) and neglecting the higher orders of η/c^2 , η'/c^2 and η/ac , (37) gives —

$$A(\omega) = \frac{a}{\rho C_2} \frac{1}{[\omega - (-iA_2 - B_2)] [\omega - (-iA_2 + B_2)]} \tag{38}$$

where,

$$C_2 = 1 + \frac{4\eta}{ac\rho} - \frac{2Gh^2}{ac^3\rho}$$

$$A_2 = \frac{2}{\rho C_2} \left(\frac{G}{ac} + \frac{\eta}{a^2} \right)$$

$$B_2 = \frac{2\sqrt{(G\rho)}}{a\rho C_2} \sqrt{\left[1 - \frac{G}{c^2\rho} + \frac{2\eta}{ac\rho} - \frac{2Gh^2}{ac^3\rho} \right]}$$

It may be noted that both A_2 and B_2 are positive since the negative quantities are small.

Case I

For the pressure prescribed in (12) the displacement potential, after performing integration as earlier, is given by (for $\tau > 0$)—

$$\begin{aligned} \phi &= \frac{P_0 a}{\rho r C_2 [(A_2 - \lambda_0)^2 + B_2^2]} \left[-\exp \{ -\delta_1 (r - a) - \lambda_0 \tau \} + \frac{\sqrt{\{(A_2 - \lambda_0)^2 + B_2^2\}}}{B_2} \times \right. \\ &\quad \left. \times \exp \left(-A_2 \left\{ \tau + \frac{[B_2^2 - A_2^2]h^2}{2A_2 c^2} [r - a] \right\} \right) \sin \left(B_2 \tau - \frac{A_2 B_2 h^2}{c^3} [r - a] + \tan^{-1} \frac{B_2}{A_2 - \lambda_0} \right) \right] \\ &= \frac{P_0 a}{\rho r C_2 [(A_2 - \lambda_0)^2 + B_2^2]} \left[-\exp \{ -\delta_1 (r - a) - \lambda_0 \tau \} + \right. \\ &\quad \left. + \frac{\sqrt{\{(A_2 - \lambda_0)^2 + B_2^2\}}}{B_2} \exp D_2 \sin \left\{ F_2 + \tan^{-1} \frac{B_2}{A_2 - \lambda_0} \right\} \right] , \text{ for } \tau > 0 \tag{39} \end{aligned}$$

and $\phi = 0$, for $\tau < 0$

where,

$$\delta_1 = \frac{h^2 \lambda_0^2}{2c^3}$$

$$D_2 = -A_2 \left[t - \left\{ \frac{1}{c} - \frac{h^2}{2c^3 A_2} (B_2^2 - A_2^2) \right\} (r - a) \right]$$

$$F_2 = B_2 \left[t - \left\{ \frac{1}{c} + \frac{h^2 A_2}{c^3} \right\} (r - a) \right]$$

The corresponding displacement is given by (for $\tau > 0$) —

$$\begin{aligned}
 u_r = & \frac{P_0 a}{\rho \{(A_2 - \lambda_0)^2 + B_2^2\}} \left[\frac{1}{r^2} \exp \{ -\delta_1 (r - a) - \lambda_0 \tau \} - \right. \\
 & - \frac{1}{r^2} \frac{\sqrt{\{(A_2 - \lambda_0)^2 + B_2^2\}}}{B_2} \exp D_2 \sin \left(F_2 + \tan^{-1} \frac{B_2}{A_2} \right) - \\
 & - \frac{1}{r} \left(-\delta_1 + \frac{\lambda_0}{c} \right) \exp \{ -\delta_1 (r - a) - \lambda_0 \tau \} + \\
 & + \frac{1}{r} \frac{\sqrt{\{(A_2 - \lambda_0)^2 + B_2^2\}}}{B_2 c} \sqrt{\left\{ B_2^2 \left(1 + \frac{h^2 A_2}{c^2} \right)^2 + A_2 \left(1 - \frac{h^2}{2c^2 A_2} [B_2^2 - A_2^2] \right)^2 \right\}} \times \\
 & \times \exp D_2 \sin \left\{ F_2 + \tan^{-1} \frac{B_2}{A_2 - \lambda_0} - \tan^{-1} \frac{B_2 \left(1 - \frac{h^2 A_2}{c^2} \right)}{A_2 \left(1 - \frac{h^2}{2c^2 A_2} [B_2^2 - A_2^2] \right)} \right\} \left. \right] \quad (40)
 \end{aligned}$$

In elastic case, $\eta = \eta' = 0$, $G = \mu$ and $G' = k$, so that $h^2 = 0$ and the displacement potential ϕ in (38) becomes (for $\tau > 0$) —

$$\phi = \frac{P_0 a}{\rho r [(A_2 - \lambda_0)^2 + B_2^2]} \left[-\exp(-\lambda_0 \tau) + \frac{\sqrt{\{(A_2 - \lambda_0)^2 + B_2^2\}}}{B_2} \exp(-A_2 \tau) \cos \left\{ F_2 - \tan^{-1} \frac{A_2 - \lambda_0}{B_2} \right\} \right] \quad (41)$$

where angular frequency,

$$\begin{aligned}
 B_2 &= \frac{2\sqrt{(\mu\rho)}}{a\rho} \sqrt{\left(1 - \frac{\mu}{c^2\rho} \right)} = \frac{2}{a} \sqrt{\frac{\mu}{\rho}} \sqrt{\left(1 - \frac{\mu}{\lambda + 2\mu} \right)} \\
 &= \frac{c}{a} \frac{\sqrt{(1 - 2\bar{\sigma})}}{1 - \bar{\sigma}}, \quad c^2 = \frac{1}{\rho} (\lambda + 2\mu)
 \end{aligned}$$

and damping is —

$$A_2 = \frac{2\mu}{ac\rho} = \frac{c}{a} \frac{1 - 2\bar{\sigma}}{1 - \bar{\sigma}}.$$

The expression (40) for the displacement potential ϕ is same as (23) and also similar to that obtained by Blanke (1952).

The response for step function of pressure at the surface of the cavity can be obviously obtained by substituting $\lambda_0 = 0$ in (39) and (40). The corresponding response in elastic case obtained by Jeffreys (1931) may also be easily deduced putting $\eta = \eta' = 0$, $G = \mu$ and $G' = k$.

Case II

For the pressure prescribed in (14) we have,

$$\phi = \frac{P_0 a}{\rho r C_2 [(A_2 - \lambda_0)^2 + B_2^2]} \left[\left\{ \tau - \frac{\lambda_0 h^2}{c^3} (r - a) - \frac{2(A_2 - \lambda_0)}{(A_2 - \lambda_0)^2 + B_2^2} \right\} \exp \{ -\lambda_0 \tau + \delta(r - a) \} - \frac{1}{B_2} \sin \left\{ F_2 - \tan^{-1} \frac{2B_2(A_2 - \lambda_0)}{-(A_2 - \lambda_0)^2 + B_2^2} \right\} \exp D_2 \right], \text{ for } \tau > 0 \tag{42}$$

and

$$\phi = 0, \text{ for } \tau < 0.$$

The corresponding displacement is given by (for $\tau > 0$) —

$$\begin{aligned} u_r = & \frac{P_0 a}{\rho C_2 [(A_2 - \lambda_0)^2 + B_2^2]} \left[-\frac{1}{r^2} \left\{ \tau - \frac{\lambda_0 h^2}{c^3} (r - a) - \frac{2(A_2 - \lambda_0)}{(A_2 - \lambda_0)^2 + B_2^2} \right\} \exp \{ -\lambda_0 \tau + \delta(r - a) \} \right. \\ & + \frac{1}{r^2 B_1} \sin \left\{ F_2 - \tan^{-1} \frac{2B_2(A_2 - \lambda_0)}{-(A_2 - \lambda_0)^2 + B_2^2} \right\} \exp D_2 + \\ & + \frac{1}{r} \left\{ -\left(\frac{1}{c} + \frac{\lambda_0 h^2}{2c^3} \right) + \left(\frac{\lambda_0}{c} + \frac{\lambda_0^2 h^2}{2c^2} \right) \left(\tau - \frac{\lambda_0 h^2}{c^3} [r - a] - \frac{2[A_2 - \lambda_0]}{[A_2 - \lambda_0]^2 + B_2^2} \right) \right\} \times \\ & \times \exp \{ -\lambda_0 \tau - \delta(r - a) \} + \frac{1}{B_2 r c} \sqrt{ \left\{ B_2^2 \left(1 + \frac{h^2 A_2}{c^2} \right)^2 + A_2^2 \left(1 - \frac{h^2}{2c^2 A_2} [B_2^2 - A_2^2] \right)^2 \right\} } \times \\ & \times \exp D_2 \sin \left[F_2 - \tan^{-1} \frac{2B_2(A_2 - \lambda_0)}{-(A_2 - \lambda_0)^2 + B_2^2} - \tan^{-1} \frac{B_2 \left(1 + \frac{h^2 A_2}{c^2} \right)}{A_2 \left(1 - \frac{h^2}{2c^2 A_2} [B_2^2 - A_2^2] \right)} \right] \end{aligned} \tag{43}$$

Case III

For the pressure prescribed in (16) we have—

$$\phi = \frac{P_0 a}{\rho r B_2 C_2} \exp D_2 \sin F_2, \text{ for } \tau > 0 \tag{44}$$

$$\phi = 0, \text{ for } \tau < 0$$

and the corresponding displacement is (for $\tau > 0$) —

$$\begin{aligned} u_r = & \frac{P_0 a}{\rho B_2 C_2} \exp D_2 \left[-\frac{\sin F_2}{r^2} + \frac{1}{rc} \sqrt{ \left\{ A_2^2 \left(1 - \frac{h^2}{2c^2 A_2} [B_2^2 - A_2^2] \right)^2 + B_2^2 \left(1 + \frac{h^2 A_2}{c^2} \right)^2 \right\} } \times \right. \\ & \left. \times \sin \left(F_2 - \tan^{-1} \left\{ B_2 \left(1 - \frac{h^2 A_2}{c^2} \right) / A_2 \left(1 - \frac{h^2}{2c^2 A_2} [B_2^2 - A_2^2] \right) \right\} \right) \right] \end{aligned} \tag{45}$$

5. Summary and Discussion

The integral form of the displacement potential in a general viscoelastic medium is given by (8) and (11) together. For any arbitrary time dependent pressure on the surface of a cavity, inversion of the Fourier transformation in equation (8) may be performed after choosing a particular model to be investigated. If the exact inversion is not possible it can be evaluated by numerical integration. The theory developed is particularly important in deriving the viscoelastic behaviour of the material when the type of source and displacements are known.

In further analysis three types of pressures have been considered for which displacement potentials are given in (13), (15) and (17). For Maxwell medium displacements u_r have been derived in (22), (32) and (34) where as that for Voigt medium in (40), (43) and (45) respectively. The displacement u_r in each of these media consists of two terms—one depending on $1/r$ and other on $1/r$. For pressures of cases I and II each of these terms again consists of two parts—one is non-oscillatory and other oscillatory; the non-oscillatory terms contain the exponential decay of the Forcide function; these terms decreases exponentially with distance. In case III only oscillatory terms are present.

It may be mentioned that the pressures of cases I and II may be considered in a generalised form as—

$$P(t) = \sum_{i=0}^L \sum_{j=0}^M P_{ij} t^{ni} \exp(-\lambda_j t) \quad , \quad t > 0$$

$$= 0 \quad , \quad t < 0$$

and the displacement may be obtained in a similar manner.

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