MAUSAM, **66**, 4 (October 2015), 785-792

551.510.42 : 551.551.8

Analytical and numerical solutions of crosswind integrated concentration by using different eddy diffusivities methods

M. ABDEL-WAHAB, KHALED S. M. ESSA*, M. EMBABY* and SAWSAN E. M. ELSAID*

Meteorology and Astronomy Department, Faculty of Science, Cairo, University

** Mathematics and Theoretical Physics, NRC, Atomic Energy Authority, Cairo-Egypt*

(*Received 31 July 2012, Modified 18 April 2015*)

e mail : mohamedksm56@yahoo.com

सार – इस शोध पत्र का मुख्य उद्देश्य लैपलैस ट्रान्सफोर्म (आकृति) तथा एडोमियन अपचयन विधियों का उपयोग करते हुए वायुमंडलीय विसरण समीकरण (ए डी ई) के द्वारा वायु प्रदूषण की सघनता की गणना करना है। इसका समाधान भंवर विसरणशीलता प्रोफाइल (K) और उत्सर्जन स्थान की पवन गति (u) पर निर्भर करता है। हमने लैपलैस ट्रान्सफॉर्म विधि का उपयोग करते हुए वायुमंडलीय विसरण समीकरण (ए डी ई) दो डायमेन्शस में विश्लेषित करके निकाला है और इसे लैपलास विश्लेषण के विपरीत पाया है तथा एडोमियन अपचयन विधि का प्रयोग करते हुए इसका संख्यात्मक रूप में निष्कर्ष निकाला है, अंत में हमने अपने परिणामों की तुलना प्रेक्षित डेटा से की है।

ABSTRACT. The objective of this paper is to calculate the concentration of air pollution, by solving the Atmospheric Diffusion Equation (ADE) using Laplace transform and Adomian decomposition methods. The solution depends on eddy diffusivity profile (K) and wind speed at the released point (u). We solve the ADE analytically in two dimensions using Laplace transform method and get the inversion of Laplace analytically and solving it numerically using Adomian decomposition method, then, compared our results with observed data.

Key words – Crosswind integrated concentration, Eddy diffusivities methods.

1. Introduction

Analytical solution for the Eulerian and Lagrangian particle models are usually obtained just for stationary conditions and by assuming strong assumptions on the wind speed profiles and turbulent parameters. In the analytical solutions of the diffusion - advection equation, authors assumed constant wind velocity along the whole Planetary Boundary Layer (PBL) or following a power law wind velocity (van Ulden and Hotslag, 1978; Pasquill and Smith, 1983; Seinfeld, 1986; Tirabassi *et al*., 1986; Sharan *et al*., 1996). The advection and diffusion of emitted pollutants from area sources are one of very important problems because of bearing its direct effect on calculating dispersion of containment of urban area. Air dispersion model based on its analytical solution had several advantages over numerical models because all of the parameters are explicitly expressed in mathematical form. Where the mathematical techniques can properly predict dispersion and transport of atmospheric pollutants are an essential element in the development of warning and control strategies, proper forecast of atmospheric boundary - layer height and its vertical mean wind speed which provide a basis for predictions of air concentrations under meteorological conditions that vary horizontally and vertically.

Analytical solutions are useful in examining the accuracy and performance of the numerical models through studies of the analytical solution that allows valuable insight to be gained regarding the behaviour of a system [Essa and El-Otaify, (2005)].

The Adomian decomposition method (ADM) has been applied to wide class of stochastic and deterministic problems in many interesting mathematics and physics areas (Adomian, 1994). Adomain gave a review of the decomposition method in (Adomain, 1988). Wazwaz (2001) found the numerical solution of sixth order boundary value problem by ADM, Abdel - Aziz and El-Sayed (2003) compared between Adomians decomposition method and wavelet - Galerkin method for solving integral - differential equations. El-Gamel (2007) compared between the Sine - Galerkin and the modified decomposition methods for two - point boundary - value problems.

Here, advection diffusion equation is solved in two dimensional space (x, z) using Laplace transform and Adomian decomposition method to obtain the normalized crosswind integrated concentration employing analytical and numerical forms respectively. Two models forms of the eddy diffusivities as well as the wind speed at the released point were used in the solution. Two calculated models were compared with observed data measured at Copenhagen in Denmark by using statistical technique.

2. Analytical method

The atmospheric advection - diffusion equation is on the form (Essa and El-Otaify, 2005):

$$k\frac{\partial^2 c_y(x,z)}{\partial z^2}=u\frac{\partial c_y(x,z)}{\partial x} \tag{1}$$

Equation (1) is subjected to the following boundary condition.

(a). It is assumed that the pollutants are absorbed at the ground surface, *i.e.*,

$$k\frac{\partial c_y(x,z)}{\partial z}=-v_g\, c_y(x,z) \qquad \text{at } z=0 \qquad (i)$$

where, v_g is the deposition velocity (m/s).

(b). The flux at the top of the mixing layer can be given by:

$$k\frac{\partial c(x,z)}{\partial z}=0 \qquad \text{at } z=h \qquad (ii)$$

(c). The mass continuity is written in the form:

$$u\, c_y(x,z) = Q\,\delta(z-h) \text{ at } x=0 \qquad (iii)$$

where, δ is Dirac delta function, Q is the source strength and h is mixing height.

(d). The concentration of the pollutant tends to zero at large distance of the source, *i.e.*,

$$c_y(x,z)=0 \quad \text{at } z=\infty \qquad (iv)$$

Applying the Laplace transform on equation (1) to have:

$$\left(\frac{\partial^2}{\partial z^2}-\frac{us}{k}\right)\tilde{c}_y(s,z)=-\frac{u}{k}c_y(0,z) \tag{2}$$

Substituting from equation (iii) in equation (2), we obtain that:

$$\left(\frac{\partial^2}{\partial z^2}-\frac{us}{k}\right)\tilde{c}_y(s,z)=-\frac{Q}{k}\delta(z-h) \tag{3}$$

where, $\breve{c}_y(s,z) = L_P\{\, cy(x,z)\ ;\ x\rightarrow s\}$ and L_P is the operator of the Laplace transform

$$L\left[\frac{\partial c_y(x,z)}{\partial x}\right]=s\left[\tilde{c}_y(s,z)\right]-c_y(0,z)$$

The non-homogeneous partial differential equation has a solution on the from:

$$\tilde{c}_y(s,z)=c_1\, e^{z\sqrt{\frac{su}{k}}}+c_2\, e^{-z\sqrt{\frac{su}{k}}}+\frac{1}{h\sqrt{suk}}\left[1-e^{-h\sqrt{\frac{su}{k}}}\right] \tag{4}$$

From the boundary condition (iv), we find $c_1 = 0$.

$$\tilde{c}_y(s,z)=c_2\, e^{-z\sqrt{\frac{su}{k}}}+\frac{1}{h\sqrt{suk}}\left[1-e^{-h\sqrt{\frac{su}{k}}}\right] \tag{5}$$

Using the boundary condition (iii) after taking Laplace transform, we get:

$$c_2=\frac{Q}{ks}\delta(z-h) \tag{6}$$

Substituting from equation (6) in equation (5), we get:

$$\tilde{c}_y(s,z)=\frac{Q}{ks}\delta(z-h)e^{-z\sqrt{\frac{su}{k}}}+\frac{1}{h\sqrt{suk}}\left[1-e^{-h\sqrt{\frac{su}{k}}}\right] \tag{7}$$

Taking the inverse Laplace transform for the eqn. (7), we get the crosswind integrated concentration in the form:

$$\frac{c_y(x,z)}{Q}=\left[\frac{h\sqrt{u}}{2\sqrt{\pi k^3 x^3}}-\frac{1}{h\sqrt{\pi xuk}}\right]e^{\frac{h^2 u}{4kx}}+\frac{1}{h\sqrt{\pi xuk}} \tag{8}$$

Numerical method

$$\frac{\partial^2 c_y(x,z)}{\partial z^2}=\frac{u}{k}\frac{\partial c_y(x,z)}{\partial x}=A\frac{\partial c_y(x,z)}{\partial x} \tag{9}$$

where, $A = \frac{u}{k}$

Equation (9) can be solved using Adomian decompositions method as follows:

$$L_{zz}\, c_y(x,z) = A L_x\, c_y(x,z)$$

$$\text{where, } L_{zz} = \frac{\partial^2}{\partial z^2}, L_x = \frac{\partial}{\partial x} \quad (10)$$

Multiplying both sides of this equation by L_{zz}^{-1} (inverse)

$$c_y(x,z) = c_0 + A L_{zz}^{-1} c_y(x,z)$$

$$L_{zz}^{-1} = \int_0^z \int_0^z \left[c_0 + A L_{zz}^{-1} L_x c_y(x,z)\right] dzdz \quad (11)$$

Assuming that:

$$Co = \text{M}(x) + z\,N(x) \quad (12)$$

where, M and N are unknown function which will be determined from boundary condition using equation (12) to get the general solution in the from:

$$c_{n+1} = A \int_0^z \int_0^z \frac{\partial c_n}{\partial x} dzdz \quad (13)$$

Put $n = 0$

$$c_1 = A \frac{\partial M}{\partial x} \frac{z^2}{2!} + A \frac{\partial N}{\partial x} \frac{z^2}{3!} \quad (14)$$

Assuming the solution has the form:

$$W_n = \sum_{n=0}^{\infty} c_n$$

$$W_1 = c_0 + c_1 = M(x) + zN(x) = A \frac{\partial M}{\partial x} \frac{z^2}{2!} + A \frac{\partial N}{\partial x} \frac{z^2}{3!} \quad (15)$$

By differentiating the equation (15) with respect to z and multiplying by k_z, we obtain:

$$k_z \frac{\partial W_1}{\partial z} = k_z N(x) + Azk_z \frac{\partial M}{\partial x} + A \frac{z^2}{2!} k_z \frac{\partial N}{\partial x} \quad (16)$$

TABLE 1

Estimates of the power (p) in urban areas for six stability classes based on information by Irwin (1979)

Stability Classes	Very Unstable (A)	Moderately Unstable (B)	Slightly unstable (C)	Neutral (D)	Slightly stable (E)	Moderately Stable (F)
Urban p	0.19	0.21	0.32	0.30	0.36	0.46

Using the boundary condition (*i*) at $z = 0$, we obtain

$$k_z \frac{\partial W_1}{\partial z} = k_z N(x) = -v_g\, M(x)$$

$$N(x) = \frac{-v_g}{k_z} M(x) \Rightarrow M(x) = -\frac{k_z}{v_g} N(x) \quad (17)$$

Using the boundary condition (*ii*) at $z = h$, we obtain that:

$$\frac{dN}{N(x)} = \left[\frac{2(k_0\, Ah - v_g)}{Ah(hv_g - 2k)}\right] dx \quad (18)$$

Integrating equation (18) from 0 to x, we obtain:

$$N(x) = N_0(x) e^{\left[\frac{2(k_0\, Ah - v_g)}{Ah(hv_g - 2k)}\right] x} \quad (19)$$

Using the boundary condition (*iii*), we get:

$$N_0(x) = \frac{Q}{u} \delta(z - h)$$

Substituting $N_0(x)$ in equation (19), to have:

$$N(x) = \frac{Q}{u} \delta(z - h) e^{\frac{2(Ahk_0\, u* - v_g)x}{Ah(hv_g - 2k)}} \quad (20)$$

Substituting equations (18) and (19) in equation (12), we obtains:

$$c_0 = \frac{-k(x)}{v_g} - z\,N(x) = -\left[\frac{k(x)}{v_g} + z\,N(x)\right]$$

$$= -(B + z) N(x) \quad (21)$$

where, $B = k / v_g$

TABLE 2

Values of wind speed at 10 m and 115 m and downwind distance through unstable and neutral stabilities at northern part of Copenhagen

Run no	Stability	u_{10} (m/s)	U_{115} (m/s)	Distance (x) (m)
1	Very unstable (A)	2.1	3.34	1900
1	Very unstable (A)	2.1	3.34	3700
2	Slightly unstable (C)	4.9	10.71	2100
2	Slightly unstable (C)	4.9	10.71	4200
3	Moderately unstable (B)	2.4	4.01	1900
3	Moderately unstable (B)	2.4	4.01	3700
3	Moderately unstable (B)	2.4	4.01	5400
5	Slightly unstable (C)	3.1	4.93	2100
5	Slightly unstable (C)	3.1	4.93	4200
5	Slightly unstable (C)	3.1	4.93	6100
6	Slightly unstable (C)	7.2	11.45	2000
6	Slightly unstable (C)	7.2	11.45	4200
6	Slightly unstable (C)	7.2	11.45	5900
7	Moderately unstable (B)	4.1	6.85	2000
7	Moderately unstable (B)	4.1	6.85	4100
7	Moderately unstable (B)	4.1	6.85	5300
8	Neutral (D)	4.2	8.74	1900
8	Neutral (D)	4.2	8.74	3600
8	Neutral (D)	4.2	8.74	5300
9	Slightly unstable (C)	5.1	11.14	2100
9	Slightly unstable (C)	5.1	11.14	4200
9	Slightly unstable (C)	5.1	11.14	6000

$$\frac{dN}{dx} = N\left[\frac{2\left(Ahk_0 u * - v_g\right)}{Ah\left(hv_g - 2k\right)}\right] \tag{22}$$

$$M = \frac{-k}{v_g}\frac{\partial N}{\partial x}$$

$$\frac{dM}{dx} = -\frac{k}{v_g} N \frac{2\left(Ahk_0 u * - v_g\right)x}{Ah\left(hv_g - 2k\right)} \tag{23}$$

Substituting equations (19) and (23) into equation (14), we obtain that:

$$c_1 = (AD)\left[\frac{z^3}{3!} - \frac{k}{v_g}\frac{z^2}{2!}\right]N \tag{24}$$

where,

$$D = -\frac{\left(2hAk_0 - 2v_g\right)x}{hA\left(hAv_g - 2k\right)}$$

Similarity, we get

$$c_2 = (AD)^2\left[\frac{z^5}{5!} - \frac{k}{v_g}\frac{z^4}{4!}\right]N$$

$$c_3 = (AD)^3\left[\frac{z^7}{7!} - \frac{k}{v_g}\frac{z^6}{6!}\right]N$$

$$c_4 = (AD)^4\left[\frac{z^9}{9!} - \frac{k}{v_g}\frac{z^8}{8!}\right]N \tag{25}$$

The general solution:

$$\frac{c_y(x,z)}{Q} = \frac{v_g}{u\left(hv_g - k\right)} e^{\frac{2\left[Ah\frac{dk}{dx} - v_g\right]x}{Ah\left(hv_g - 2x\right)}}$$

$$\sum_{i=1}^{n}\left[\frac{2u\left[Ah\frac{dk}{dx} - v_g\right]x}{Ahk\left(hv_g - 2x\right)}\right]^{i}\left[\frac{-kz^{2i}}{v_g(2i)!} + \frac{z^{2i+1}}{(2i+1)!}\right] \tag{26}$$

We can obtain the wind speed at source height 115 m as follows (Hanna *et al*., 1982).

$$u_{115} = u_{10}\left(\frac{z}{10}\right)^p \tag{27}$$

where,

U_{115} is the wind speed at 115 m,

U_{10} is the wind speed at 10 m height,

z is the physical height and

TABLE 3

Comparison between Observed, and different analytical, numerical normalized crosswind-integrated concentrations Cy/Q (10^{-4} sm^{-3})

Run no.	Stability	Down distance (m)	C_y/Q *10^{-4} (s/m^3)				
			Analytical model 1	Analytical model 2	Numerical model 1	Numerical model 2	Observed
1	Very unstable (A)	1900	4.48	8.95	3.59	2.08	6.48
1	Very unstable (A)	3700	3.37	4.64	4.93	3.79	2.31
2	Slightly unstable (C)	2100	1.29	6.28	7.36	4.03	5.38
2	Slightly unstable (C)	4200	1.02	3.14	2.04	1.27	2.95
3	Moderately unstable (B)	1900	5.08	10.92	1.05	1.32	8.2
3	Moderately unstable (B)	3700	3.17	6.30	8.94	3.40	6.22
3	Moderately unstable (B)	5400	1.80	8.30	1.20	6.25	4.3
5	Slightly unstable (C)	2100	4.64	9.47	1.18	3.55	6.72
5	Slightly unstable (C)	4200	1.80	9.01	1.69	8.75	5.84
5	Slightly unstable (C)	6100	0.91	12.19	3.76	1.53	4.97
6	Slightly unstable (C)	2000	1.56	5.30	2.02	2.83	3.96
6	Slightly unstable (C)	4200	0.98	2.53	1.44	7.24	2.22
6	Slightly unstable (C)	5900	0.60	1.98	5.31	1.18	1.83
7	Moderately unstable (B)	2000	2.12	8.11	1.81	2.63	6.7
7	Moderately unstable (B)	4100	1.64	3.96	1.46	6.09	3.25
7	Moderately unstable (B)	5300	1.33	3.06	1.01	8.62	2.23
8	Neutral (D)	1900	2.83	10.31	5.14	7.11	4.16
8	Neutral (D)	3600	1.30	5.45	9.14	1.50	2.02
8	Neutral (D)	5300	0.66	4.37	4.32	2.42	1.52
9	Slightly unstable (C)	2100	1.22	6.86	5.97	3.50	4.58
9	Slightly unstable (C)	4200	0.94	3.43	1.05	7.70	3.11
9	Slightly unstable (C)	6000	0.72	2.40	1.60	1.18	2.59

p is a parameter estimated by Irwin (1979), which is related to stability classes, is given in Table 1.

In the present model we used two methods for the calculation of the eddy diffusivity depends on the downwind distance (x). The first method k takes in the from $k_1(x) = 0.04\ ux$ and in the second method are referenced to where k takes in the form:

$$k_z(x) = 0.16\left(\frac{\sigma_w^2}{u}\right)x$$

σ_w is the standard deviation vertical velocity (Arya, 1999).

The data set used was observed from the atmospheric diffusion experiments conducted at the northern part of Copenhagen, Denmark, under unstable conditions (Gryning and Lyck, 1984; Gryning *et al*., 1987). The tracer sulfur hexafluoride (SF_6) was released from a tower at a height of 115 m without buoyancy. The values of different parameters such as stability, wind speed at 10 m (U_{10}), wind speed at 115 m (U_{115}), and downwind distance during the experiment are represented in Table 2.

Comparison between analytical model 1, 2 and observed normalized crosswind integrated concentration shows that analytical model 2 agrees with observed data than analytical model 1 (Table 3). Comparison between numerical model 1, 2 and observed normalized crosswind integrated concentration shows that numerical model 1 agrees with observed data than numerical model 2.

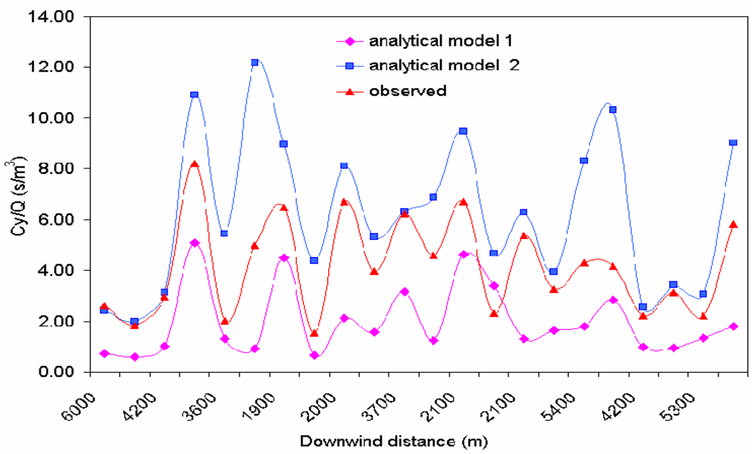

Fig. 1. Comparison between analytical, observed normalized crosswind integrated concentration and downwind distance

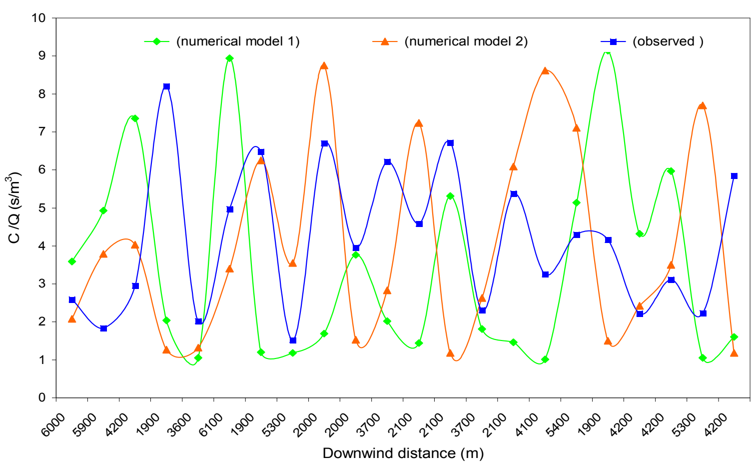

Fig. 2. Comparison between numerical cross wind integrated concentration and downwind distance

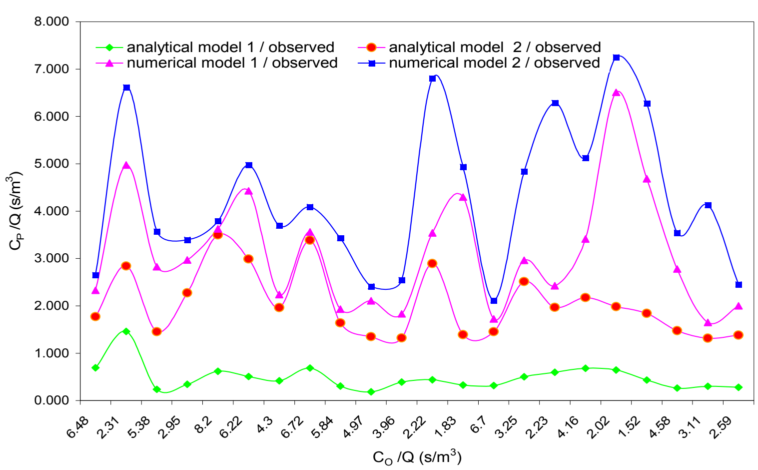

Fig. 3. Ratio of predicted and observed normalized crosswind concentrations via observed normalized crosswind concentrations for all model

TABLE 4

Comparison between different models according to standard statistical performance measure

Models	NMSE	FB	COR	FAC2
Analytical model 1	0.79	0.71	0.71	0.48
Analytical model 2	0.30	- 0.40	0.78	1.56
Numerical model 1	0.66	0.04	- 0.11	1.19
Numerical model 2	0.79	0.19	- 0.08	1.09

Fig. 1 shows that the variation of analytical and observed normalized crosswind concentration data with down distance. Fig. 2 shows that the variation of numerical and observed normalized crosswind concentrations data downwind distance. One find that analytical model (1) and (2) and numerical model (1) have points agreeing with the observed data, while the others points are over predicted.

Fig. 3 shows that analytical model (1) under predicated with observed data, while analytical model (2) have most points within factor of two with observed data. On other hand numerical model 1 has most data within a factor of 2 (FAC2). While numerical model (2) has most points over predicted with observed data.

3. Statistical method

Now, the statistical method and comparison among analytical, statically and observed results will be present (Hanna, 1989). The following standard statistical performance measures characterize the agreement between model prediction ($C_p = C_{pred}/Q$) and observations ($Co = C_{obs}/Q$) (Table 4):

$$\text{Normalized mean square error (NMSE)} = \frac{\overline{\left(C_p - C_o\right)}}{\left(\overline{C_p}\,\overline{C_o}\right)}$$

$$\text{Fractional Bias (FB)} = \frac{\left(\overline{C_o} - \overline{C_p}\right)}{\left[0.5\left(\overline{C_o} + \overline{C_p}\right)\right]}$$

Correlation Coefficient

$$\text{(COR)} = \frac{1}{N_m}\sum_{i=1}^{N_m}\left(C_{pi} - \overline{C_p}\right)\times\frac{\left(C_{oi} - \overline{C_o}\right)}{\sigma_p\,\sigma_o}$$

$$\text{Factor of two (FAC2)} = 0.5 \le \frac{C_p}{C_o} \le 2.0$$

where, σ_p and σ_o are the standard deviations of C_p and C_o respectively. Here the over bars indicate the average over all measurements (Nm). A perfect model would have the following idealized performance:

NMSE = FB = 0 and COR = FAC2 = 1.0

From the statistical analysis, we find that the four models are within a factor of 2 with observed data. Regarding NMSE, the analytical models (1), (2) and numerical model (1) are better than numerical model (2). The analytical model (2) and numerical model (1) are also the best regarding FB.

4. Conclusions

We have used an analytical and numerical solution of two-dimensional atmospheric diffusion equation by Laplace transform and Adomian decomposition methods respectively to calculate normalized crosswind concentrations for continuous emission of sulfur hexafluoride (SF_6). In this model the vertical eddy diffusivity depends on the downwind distance and is calculated using two methods ($k_1 = 0.04\ ux$ and $k_2 = 0.16(\sigma_w^2/u)\ x$) it is observed that analytical model 2 and numerical model 1 have most points within a factor of two with the observed data. The other two models over predicted. From the statistical analysis, we find that the four models are within a factor of 2 (FAC2) with observed data, regarding NMSE, the analytical model 2 and numerical model 1 are better than the other model. Also the analytical model 2 and numerical model 1 are the best regarding FB. The correlation of analytical model 1 and analytical model 2 are 0.71 and 0.78 respectively which are stronger with the observed data than the correlation of numerical model 1 which equals 0.11.

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