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Study Gaussian plume model and the Gradient Transport (K) of the advection-diffusion equation and its applications

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सार — इस शोध पत्र में, संवहन-विसरण समीकरण (ADE) के किसी एक आयाम कालाश्रित और स्थिर अवस्था के तीन आयामों को वायुमंडलीय सीमा परत (ABL) में सांद्रता का अनुमान लगाने के लिए विश्लेषणात्मक रूप से हल किया गया है, इस अवधारणा को ध्यान में रखते हुए कि ABL ऊंचाई (h) को उप-परतों में विभाजित किया गया है और डाउनविंड दूरी को भी अंतराल में प्रत्येक आयताकार क्षेत्र में विभाजित कियागया है, पवन की गति और भंवर प्रसार के औसत मान को ध्यान में रखकर लॉप्लास ट्रांसफ़ॉर्म विधि का उपयोग करके ADE आकलित किया गया है। प्रस्तावित मॉडल, गॉसियन प्लूम मॉडल और पूर्ववर्ती शोध कार्य (Essa *et al*., 2019) की तुलना आयोडीन-135 की प्रेक्षित सांद्रता से की गई है जिसे मिस्र के परमाणु ऊर्जा प्राधिकरण, परमाणु अनुसंधान रिएक्टर, इंशास, काहिरा, मिस्र में मापा गया था। सांख्यिकीय विश्लेषण से पता चलता है कि सांद्रता के प्रस्तावित और प्रायोगिक मूल्यों के बीच अच्छा संबंध है।

ABSTRACT. In this paper, one dimension time-dependent and the steady state three dimensions of an advection-diffusion equation (ADE) has been solved analytically. To estimate the concentration into the atmospheric boundary layer (ABL) taking into account the assumption that the ABL height (h) is divided into sub-layers and the downwind distance is also divided into intervals within each rectangular area. The ADE is estimated by using the Laplace transform method assuming that the mean values of wind speed and eddy diffusivity. The proposed model, Gaussian plume model and previous work (Essa *et al*., 2019) was compared with the observed concentration of Iodine-135 which was measured at the Egyptian Atomic Energy Authority, Nuclear Research Reactor, Inshas, Cairo, Egypt. The statistical analysis shows that there is a good agreement between the proposed and experimental values of concentration.

Key words – Gaussian plume model, Gradient Transport (K), Turbulent Diffusion.

1. Introduction

The classical Gaussian of diffusion models are mostly used in effecting the impacts of finding and proposed source of air contaminants on local and urban air quality Arya (1999). The lateral and vertical dispersion parameters, respectively σ_y and σ_z, represent the turbulent parameterization in this approach, once they contain the physical ingredients that describe the dispersion process and, consequently, shows the spatial extent of the contaminant plume under the effect of the turbulent motion into the Planetary boundary layer (PBL) Abdul-Wahab (2006).

The atmospheric advection-diffusion equation has long been used to know the transport of pollutants in a turbulent atmosphere was investigated by Seinfeld (1986). An analytical dispersion model for source into the atmospheric surface layer with dry deposition to the ground surface has been made by Kumar and Sharan (2016). Also, the variation of eddy diffusivity on the mimics of behavior of the advection-diffusion equation was investigated by Essa *et al*. (2018). Essa *et al.* (2020) used the advection-diffusion equation with variable vertical eddy diffusivity and wind speed by using Hankel transform to get the crosswind integrated concentration. Essa *et al*. (2021) obtained the solution of the advection-diffusion equation in three dimensions by using Hankel transform, Essa *et al*. (2022).

In this work, one dimension of time-dependent and the steady state three dimensions of the advection-diffusion equation (ADE) has been calculated analytically. To estimate the concentration into the atmospheric boundary layer (ABL) taking into account the assumption that the ABL height (h) is divided into sub-layers and the downwind distance is also divided into intervals within each rectangular area. The ADE is estimated by using the Laplace transform method assuming the mean values of wind speed and eddy diffusivity. The proposed model, Gaussian plume model and previous work (Essa *et al*., 2019) was compared with the observed concentration of Iodine-135 which was measured at the Egyptian Atomic Energy Authority, Nuclear Research Reactor, Inshas, Cairo, Egypt. The statistical analysis shows that there is a good agreement between the proposed and experimental values of concentration.

2. Turbulent diffusion

Three-dimensional turbulent motions are characterized by a rather wide range of scales, its description faces all the problems and difficulties that is associated with turbulence.

2.1. *Mean Diffusion Equation*

The diffusion equation must be solved simultaneously with the Navier-Stokes equation of motion. These equations are nonlinear and their complete numerical solution has not yet become possible for large Reynolds number flows, such Large-eddy simulations (LES) using state-of-the-art computers are probably the most sophisticated, but also, the most expensive numerical models of turbulence and diffusion [Deardorff (1972); Lamb (1982); Nieuwstatt and Walk (1987)]. The large-eddy simulations (LES) are based on analytical solutions of the mean diffusion equation that is obtained by Reynolds averaging of the instantaneous equation.

$$\frac{\partial \overline{C}}{\partial t} + \overline{u}\frac{\partial \overline{C}}{\partial x} + \overline{v}\frac{\partial \overline{C}}{\partial y} + \overline{w}\frac{\partial \overline{C}}{\partial z} = M\Delta^2 \overline{C} - \left(\frac{\partial C'u'}{\partial x} + \frac{\partial C'v'}{\partial y} + \frac{\partial C'w'}{\partial z}\right) \quad (1)$$

where, $\overline{C}$ is an average concentration, u, v and w are the components of wind speed, and M is molecular diffusion and the last three terms are turbulence mass fluxes. molecular diffusion terms can be neglected because turbulence dominates the diffusion process. The mean diffusion equation for a uniform mean flow is reduced as follows:

$$\frac{\partial \overline{C}}{\partial t} + \overline{u}\frac{\partial \overline{C}}{\partial x} + \overline{v}\frac{\partial \overline{C}}{\partial y} + \overline{w}\frac{\partial \overline{C}}{\partial z} = -\left(\frac{\partial \overline{C'u'}}{\partial x} + \frac{\partial \overline{C'v'}}{\partial y} + \frac{\partial \overline{C'w'}}{\partial z}\right) \quad (2)$$

This can be solved if the terms on the right-hand side can be specified in terms of the mean concentration field as follows:

$$c'u' = -k_x \frac{\partial \overline{C}}{\partial x}$$
$$c'v' = -k_y \frac{\partial \overline{C}}{\partial y} \quad (3)$$
$$c'w' = -k_z \frac{\partial \overline{C}}{\partial z}$$

Substituting from Eqn. (3) in Eqn. (2) yields:

$$\frac{\partial \overline{C}}{\partial t} + \overline{u}\frac{\partial \overline{C}}{\partial x} = \frac{\partial}{\partial x}\left(k_x \frac{\partial \overline{C}}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial \overline{C}}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_y \frac{\partial \overline{C}}{\partial z}\right) \quad (4)$$

2.2. *Continuous source*

For an elevated continuous point source of an effective height "H" above a reflecting ground surface, the method of images yields:

$$\overline{C}(x,y,z) = \frac{Q}{2\pi \overline{u}\sigma_y\sigma_z}\exp\left(-\frac{y^2}{2\sigma_y^2}\right)\left\{\exp\left[\frac{-(z-H)^2}{2\sigma_z^2}\right] + \exp\left[\frac{-(z+H)^2}{2\sigma_z^2}\right]\right\}e^{-\frac{vx}{u}} \quad (6)$$

where, σ_y and σ_z are the dispersion parameters in crosswind and vertical directions of the plume, Q is the emission rate, $e^{-vx/u}$ is the radioactive decay for isotope, $\upsilon = 2.9 \times 10^{-5}\ s^{-1}$ for Iodine-135.

H is the effective stack height; $H = h_s + \Delta h$, h_s is the stack height and Δh is the plume rise, u is the mean wind speed, and y, z are the crosswind and vertical coordinates, respectively.

$$H = h_s + \Delta h = h_s + 3(w/u)\, D \quad (7)$$

where, w is the exit velocity of the pollutants and D is the internal stack diameter.

The ground level concentration (G. L.C.) at the plume 's centerline is given by:

$$\overline{C}_0 = \frac{Q}{\pi \overline{u}\sigma_y\, \sigma_z}\left[\exp\left(\frac{-H^2}{2\sigma_z^2}\right)\right] \quad (8)$$

Also, the maximum concentration is:

$$C_{\max} = \frac{Q}{\pi \bar{u} e \sigma_y \sigma_z} \tag{9}$$

$$\sigma_x = \sqrt{\frac{2k_x x}{u}}, \sigma_y = \sqrt{\frac{2k_y x}{u}}, \text{ and } \sigma_z = \sqrt{\frac{2k_z x}{u}}$$

Crosswind and vertical dispersion parameters for convective conditions are taken from Lidiane *et al*. (2008) into the form:

$$\frac{\sigma_y^2}{h^2} = \frac{0.66}{\pi^2}\int_0^\infty \frac{sin^2\left(0.75\pi\Psi^{\frac{1}{3}}Xn'\right)}{n'^2\left(1+n'\right)^{\frac{5}{3}}}dn' \tag{10}$$

$$\frac{\sigma_z^2}{h^2} = \frac{0.98}{\pi^2}\int_0^\infty \frac{sin^2\left(0.98\pi\Psi^{\frac{1}{3}}Xn'\right)}{n'^2\left(1+n'\right)^{\frac{5}{3}}}dn' \tag{11}$$

where, $n' = \frac{1.5z}{u\left(f_m^*\right)_i}n$; $\left(f_m^*\right)_i$ is the reduced frequency of the convective spectral peak into the form $\left(f_m^*\right)_i = \frac{z}{h}$.

The advection diffusion equation in two dimensions can be written as follows:

$$u\frac{\partial C_y(x,z)}{\partial x} = \frac{\partial}{\partial z}\left[k_z \frac{\partial C_y(x,z)}{\partial z}\right] \tag{12}$$

where, C_y (x, z)is the crosswind integrated concentration of pollutants, u is the wind speed (m/s) and K_z is vertical eddy diffusivity (m^2/s). One can solve the advection-diffusion equation for non-homogeneous turbulence by the Laplace transform technique. A stepwise approximation has been performed on these coefficients discretizing the height h of the PBL into N sub-intervals in a manner of inside each sub-region, $k(z)$ and $u(z)$, assuming the following average values:

$$k_n = \frac{1}{z_{n+1} - z_n}\int_{z_n}^{z_{n+1}} k(z)dz$$

$$u_n = \frac{1}{z_{n+1} - z_n}\int_{z_n}^{z_{n+1}} u(z)dz$$

Then, Eqn. (12) becomes:

$$u_n = \frac{\partial c_{yn(x,z)}}{\partial x} = k_n(z)\frac{\partial c_{yn(x,z)}}{\partial z^2} \tag{13}$$

For n=1:N.

Applying the Laplace transform on "x" under the boundary conditions:

$$C_{yn}(0, z_n) = \frac{Q}{u_n}\delta(z_n - h_s) \tag{i}$$

$$k_n(z)\frac{\partial c_{yn(x,z)}}{\partial z} = 0 \qquad \text{at } z_n = 0, h \tag{ii}$$

where, h_s is a stack height and "h" is a mixing height.

Then the equation (9) can be written as:

$$\int_0^\infty u_n \frac{\partial c_{y_n}}{\partial x}e^{-sx}dx = k_n(z)\int_0^\infty \frac{\partial^2 c_{y_n}}{\partial z_n^2}e^{-sx}dx \tag{14}$$

Integrating and substituting into the equation (14), one gets:

$$-u_n c_{y_n}(0,z) + s\ c_{y_n}\tilde{c}(s,z) = k_n(z)\frac{\partial^2 \tilde{c}_{y_n}(s,z)}{\partial z_n^2} \tag{15}$$

Applying the boundary condition (i), one gets:

$$\frac{\partial^2 \tilde{c}_{y_n}(s,z)}{\partial z_n^2} - \frac{su_n}{k_n}\tilde{c}_{yn}(s,z) = -\frac{Q}{k_n}\delta(z_n - h_s) \tag{16}$$

Now applying Laplace transform on z then:

$$p^2\tilde{\tilde{c}}_{y_n}(s,p) - pc_{y_n}(s,0) - \frac{\partial \tilde{c}_{y_n}(s,0)}{\partial z} - \frac{u_n s}{k_n}\tilde{\tilde{c}}_{y_n}(s,p) = -\frac{Q}{k_n}e^{-ph_s} \tag{17}$$

Substituting condition (ii), equation (17) becomes:

$$\tilde{\tilde{c}}_{y_n}(s,p) = \frac{c_{y_n}(s,0)p}{\left(p^2 - \frac{us}{k_n}\right)} - \frac{Qe^{-ph_s}}{k_n\left(p^2 - \frac{us}{k_n}\right)} \tag{18}$$

$$\tilde{\tilde{c}}_{y_n}(s,p) = c_{y_n}(s,0)F(s,p) - \frac{Q}{k_n}e^{-ph_s}G(s,p) \qquad (19)$$

where,

$$F(s,p) = \frac{p}{\left(p^2 - \frac{us}{k_n}\right)} \text{ and } G(s,p) = \frac{1}{\left(p^2 - \frac{us}{k_n}\right)}$$

The inverse of Laplace transform on "z" is taken *i.e.*, $L^{-1}\left[\tilde{\tilde{c}}_y(s,p),z\right] = \tilde{c}_{y_n}(s,z)$

$$\tilde{c}_{y_n}(s,z) = \frac{c_{y_n}(s,0)}{2}\left(e^{\sqrt{\frac{su}{kn}}z} + e^{-\sqrt{\frac{su}{kn}}z}\right) - \frac{Q}{2k_n}\sqrt{\frac{k_n}{su}}\left[e^{\sqrt{\frac{su}{kn}}(z-h_s)} + e^{-\sqrt{\frac{su}{kn}}(z-h_s)}\right]H(z-h_s) \qquad (20)$$

Let $R_n = \sqrt{\frac{su}{k_n}}$ and $R_a = \sqrt{suk_n}$

$$\tilde{c}_{y_n}(s,z) = \frac{c_{y_n}(s,0)}{2}\left(e^{R_n z} + e^{-R_n z}\right) - \frac{Q}{2R_a}\left[e^{R_n(z-h_s)} - e^{-R_n(z-h_s)}\right]H(z-h_s) \qquad (21)$$

$$\tilde{c}_{y_n}(s,z) = c_{y_n}(s,0)\cosh R_n z - \frac{Q}{R_a}\sinh R_n(z-h_s) * H(z-h_s) \qquad (22)$$

Applying the boundary condition (ii) one gets:

$$k_n(z)\frac{\partial}{\partial z}\tilde{\tilde{c}}_{y_n}(s,z) = 0 \text{ at } z = h \text{ then: } \frac{\partial}{\partial z}\tilde{\tilde{c}}_{y_n}(s,z) = R_n c_{y_n}(s,0)\sinh R_n z - \frac{Q}{R_a}R_n\cosh R_n(z-h_s) H(z-h_s) - \frac{Q}{R_a}\sinh R_n(z-h_s)\frac{\partial}{\partial z}H(z-h_s) \qquad (23)$$

$$c_{y_n}(s,0)\sinh(R_n h) = \frac{Q}{R_a}\cosh\left[R_n(h-h_s)\right]H(h-h_s) \qquad (24)$$

$$c_{y_n}(s,0) = \frac{Q}{R_a}\frac{\cosh\left[R_n(h-h_s)\right]}{R_a\sinh(R_n h)}$$

$$c_{y_n}(s,0) = \frac{Q}{\sqrt{suk_n}}\frac{\cosh\sqrt{\frac{su}{k_n}}(h-h_s)}{\sinh\sqrt{\frac{su}{k_n}}h} \qquad (25)$$

Substituting equation (25) for equation (22) then one gets:

$$\tilde{c}_{y_n}(s,z) = \frac{Q}{\sqrt{suk_n}}\frac{cosh\sqrt{\frac{su}{k_n}}(h-h_s)}{sinh\sqrt{\frac{su}{k_n}}h}cosh R_n z - \frac{Q}{R_a}sinh R_n(z-h_s) * H(z-h_s) \qquad (26)$$

By using the Gaussian quadrature formulas, one gets the concentration in three dimensions as follows:

$$c_{y_n}(x,y,z) = \frac{Q}{\sqrt{2\pi}\sigma_y}\exp\left(\frac{-y^2}{2\sigma_y} - \frac{vx}{u}\right)\sum_{i=1}^{8} a_i\left(\frac{pi}{x}\right)\frac{1}{\sqrt{\frac{u_n k_n(z)p_i}{x}}}\frac{\cosh\sqrt{\frac{p_i u_n}{xk_n}}(z_i-h_s)\cosh(R_n z)}{\sinh\sqrt{\frac{p_i u_n}{xk_n}}z_i} \qquad (27)$$

where, $u(z) = 0.16 * \left(\frac{w_*}{u_r}\right)^2 * z^p$ and $k = 0.31 * \left(\frac{w_*}{u_r}\right)^2 * z^n$, where w_* is the convective vertical velocity and u_r is the reference wind speed.

2.3. *Results and discussion*

The observed data of I^{135} isotope concentration was obtained from dispersion as experiments conducted in unstable conditions air samples which were collected around the Egyptian Atomic Energy Authority. The vertical height is 0.7 m above ground from a stack height of 43 m, for twenty-four hours working, where the air samples were collected for half an hour at a height 0.7m with a roughness length of 0.6 cm. The values of "p" and 'n' are functions of air stability are taken from Hanna

TABLE 1

Power-law exponent "p" and 'n' are functions of air stability in urban area

	A	B	C	D	E	F
p	0.15	0.15	0.20	0.25	0.40	0.60
n	0.85	0.85	0.80	0.75	0.60	0.40

TABLE 2

Meteorological data of the nine convective test runs at Inshas site in March and May 2006

Run no.	Working hours of the source	Release rate (Bq)	Wind speed (ms^{-1})	Wind Direction(deg)	W*(ms^{-1})	P-G stability class	h (m)	Vertical distance (m)
1	48	1028571	4	301.1	2.27	A	600.85	5
2	49	1050000	4	278.7	3.05	A	801.13	10
3	1.5	42857.14	6	190.2	1.61	B	973	5
4	22	471428.6	4	197.9	1.23	C	888	5
5	23	492857.1	4	181.5	0.958	A	921	2
6	24	514285.7	4	347.3	1.3	D	443	8.0
7	28	1007143	4	330.8	1.51	C	1271	7.5
8	48.7	1043571	4	187.6	1.64	C	1842	7.5
9	48.25	1033929	4	141.7	2.1	A	1642	5.0

et al. (1982) and presented in Table 1. The meteorological data during the experiments are taken from Essa and El-Otaify (2008) and presented in Table 2. The observed concentration of I^{135} isotope, the predicted concentrations of Eqns. (6, 10, 11) and (27) below the plume 's centerline are also presented in Table 3. A comparison between predicted and observed concentrations of radioactive I^{135} *via* downwind distance in unstable conditions at Inshasis shown in Fig. 1, also, the relation between predicted and observed concentration data is shown in Fig. 2.

From the two figures, one finds that the best model is obtained from Eqn. (27) because of the vertical eddy diffusivity as a function of the power law into the vertical height than the Gaussian plume model. The predicted models achieved 99%. Also, the Gaussian model (6, 10, 11) gives a good result and achieved 79% because of the strongest of the dispersion parameters (10, 11).

2.4. *Statistical Technique of continuous source*

Comparing between the Gaussian, predicted and observed concentrations was introduced by (Hanna, 1989).

Where, NMSE is the normalized mean square error, FB is the fraction bias, COR is the correlation coefficient and FAC2 is a factor of two.

One can easily see from Table 4, the statistical technique shows that the entire proposed model inside a factor of two with observed concentration data. Also, the statistical shows that the predicted model Eqn. (27) is the best for NMSE, FB, COR and FAC2 than, the Gaussian model Eqns. (6, 10, 11) for homogeneity.

3. Instantaneous source

For an instantaneous point release at an effective height H above the reflective ground surface. The coordinates of the real and image source are (0, 0, H) and (0, 0, -H), respectively. Then the concentration field due to both the real and an image source in an infinite medium is given by:

$$c(y,z,t)=\frac{Q_{ip}}{(2\pi)^{3/2}\sigma_y\sigma_z}\exp\left(-\frac{y^2}{2\sigma_y^2}\right)\left\{\exp\left[\frac{-(z-H)^2}{2\sigma_z^2}\right]+\exp\left[\frac{-(z+H)^2}{2\sigma_z^2}\right]\right\}e^{-vt} \qquad (28)$$

TABLE 3

Observed, Gaussian and predicted concentrations for Run 9 experiments

Test	Downwind distance (m)	Observed conc.(Bq/m^3)	Gaussian conc. Eqns. (6,10,11) (Bq/m^3)	Predicted conc. Eq.(27) (Bq/m^3)
1	100	0.025	0.039975	0.027
2	98	0.037	0.03302	0.025
3	136	0.091	0.082803	0.086
4	135	0.197	0.166257	0.182
5	106	0.272	0.274148	0.271
6	186	0.188	0.107066	0.234
7	165	0.447	0.216606	0.367
8	154	0.123	0.151414	0.111
9	106	0.032	0.044721	0.041

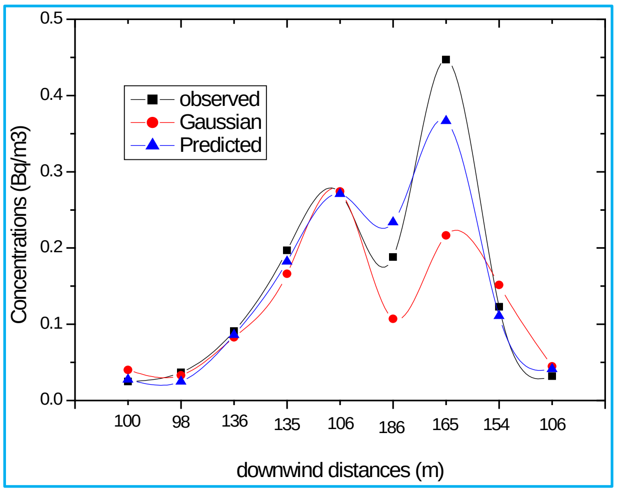

Fig. 1. Explains the relation between the Gaussian, predicted model and observed Concentrations (Bq/m^3) *via* downwind distances

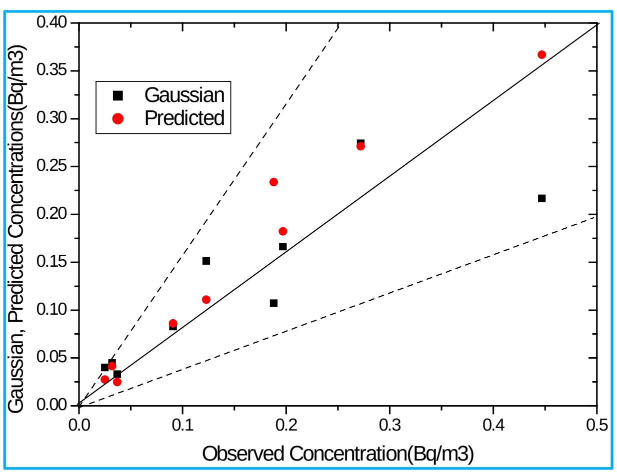

Fig. 2. Shows that the achieving between the Gaussian, predicted concentrations with observed concentration

TABLE 4

Comparison between the Gaussian, predicted and observed concentrations in unstable conditions

	NMSE	FB	COR	FAC2
Gauss. Eqn. (6, 10, 11)	0.35	0.23	0.84	0.79
Pred. Eqn. (27)	0.04	0.05	0.98	0.99

where, $e^{-\upsilon t}$ is the radioactive decay for isotope, υ=2.9 × 10^{-5} $s^{-1}$for I^{135} and the effective height "*H*" is taken the form:

$$H = h_s + \Delta h = h_s + 3(w/u)D \tag{c}$$

where, w is the exit velocity of the pollutants, and D is the internal stack diameter.

$$\sigma_x = \sqrt{2k_x t}, \sigma_y = \sqrt{2k_y t} \text{ and } \sigma_z = \sqrt{2k_z t} \tag{d}$$

where, $k_y = k_z = 0.4 u_* x$ (e)

where, u_* is the friction velocity.

The advection diffusion equation can be written as follows:

$$\frac{\partial C(t,z)}{\partial t} = \frac{\partial}{\partial z}\left[k_z \frac{\partial C(t,z)}{\partial z}\right] \tag{29}$$

where, $C(t, z)$ is the one-dimension time-dependent concentration of pollutants, and K_z is vertical eddy diffusivity that is taken as a function of the power law of vertical distance. One can solve the advection-diffusion equation for non-homogeneous turbulence by the Laplace transform technique, A stepwise approximation has been performed on these coefficients discretizing the height h of the PBL into N sub-intervals in a manner of inside each sub-region, $k(z)$ assuming the following average values:

$$k_n = \frac{1}{z_{n+1} - z_n} \int_{z_n}^{z_{n+1}} k(z)dz$$

Then, Eqn. (29) becomes:

$$\frac{\partial c_{yn(t,z)}}{\partial t} = k_n(z)\frac{\partial^2 c_{yn(t,z)}}{\partial z^2} \quad (30)$$

For n=1:N.

Applying the Laplace transform on "t" under the boundary conditions as follows:

$$C_{yn}(0, z_n) = Q\delta(z_n - h_s) \quad (a)$$

$$k_n(z)\frac{\partial c_{yn(t,z)}}{\partial z} = 0 \quad \text{at} \quad z = h \quad (b)$$

$$k_n(z)\frac{\partial c_{yn(t,z)}}{\partial z} = v_d c_{yn}(t,0) \quad \text{at} \quad z = 0 \quad (c)$$

where, "h" is a mixing height and v_d is the deposition velocity equals 0.01m/s for Iodine-135.

Then the equation (30) can be written as:

$$\int_0^\infty \frac{\partial c_{y_n}}{\partial t} e^{-lt}\, dt = k_n(z)\int_0^\infty \frac{\partial c_{y_n}}{\partial z_n^2} e^{-lt}\, dt \quad (31)$$

Integrating and substituting into the equation (31), one gets:

$$-c_{y_n}(0,z) + 1c_{y_n}\tilde{c}(s,z) = k_n(z)\frac{\partial^2 \tilde{c}_{y_n}(l,z)}{\partial z_n^2} \quad (32)$$

Applying the boundary condition (a), one gets:

$$\frac{\partial^2 \tilde{c}_{y_n}(l,z)}{\partial z_n^2} - \frac{l}{k_n}\tilde{c}_{y_n}(l,z) = -\frac{Q}{k_n}\delta(z_n - h_s) \quad (33)$$

Now applying Laplace transform on z then:

$$m^2 \tilde{\tilde{c}}_{y_n}(l,m) - pc_{y_n}(l,0) - \frac{\partial c_{y_n}(l,0)}{\partial z} - \frac{l}{k_n}\tilde{\tilde{c}}_{y_n}(l,m) = -\frac{Q}{k_n}e^{-mh_s} \quad (34)$$

Substituting condition (b), equation (34) becomes:

$$\tilde{\tilde{c}}y_{y_n}(l,m) = \frac{c_{y_n}(l,0)m}{\left(m^2 - \frac{l}{k_n}\right)} + \frac{v_d c_{y_n}(l,0)}{k_n\left(m^2 - \frac{l}{k_n}\right)} - \frac{Qe^{-mh_s}}{k_n\left(m^2 - \frac{l}{k_n}\right)} \quad (35)$$

$$\tilde{\tilde{c}}_{y_n}(l,m) = c_{y_n}(l,0)H(l,m) - \frac{Q}{k_n}e^{-mh_s}G(l,m) \quad (36)$$

where,

$$H(l,m) = \frac{m + \frac{v_d}{k_n}}{\left(m^2 - \frac{l}{k_n}\right)} \quad \text{and} \quad G(l,m) = \frac{1}{\left(m^2 - \frac{l}{k}\right)}$$

The inverse of Laplace transform on "z" is taken, *i.e.*,

$$L^{-1}\left\{\left[\tilde{\tilde{c}}_y[l,m], z\right]\right\} = \tilde{c}_{y_n}(l,z)$$

$$\tilde{c}_{y_n}(l,z) = \frac{c_{y_n}(l,0)}{2}\left(e^{\sqrt{\frac{l}{k_n}}z} + e^{-\sqrt{\frac{l}{k_n}}z}\right) + \frac{v_d c_{y_n}(l,0)}{2k_n}\left(e^{\sqrt{\frac{l}{k_n}}z} + e^{-\sqrt{\frac{l}{k_n}}z}\right) - \frac{Q}{2k_n}\sqrt{\frac{k_n}{s}}\left[e^{\sqrt{\frac{s}{k_n}}(z-h_s)} - e^{-\sqrt{\frac{s}{k_n}}(z-h_s)}\right]H(z-h_s) \quad (37)$$

Let $R_n = \sqrt{\frac{l}{k_n}}$ and $R_a = \sqrt{lk_n}$

$$\tilde{c}_{y_n}(s,z)=\frac{c_{y_n}(l,0)}{2}\left(e^{R_nz}+e^{-R_nz}\right)+\frac{v_d c_{y_n}(l,0)}{2k_n}\left(e^{R_nz}+e^{-R_nz}\right)-\frac{Q}{2R_a}\left(e^{R_n(z-h_s)}-e^{-R_n(z-h_s)}\right)H(z-h_s) \tag{38}$$

$$\tilde{c}_{y_n}(l,z)=c_{y_n}(l,0)\left(\cosh R_nz+\frac{2v_d}{k_n}\sinh R_nz\right)-\frac{Q}{R_a}\sinh R_n(z-h_s)*H(z-h_s) \tag{39}$$

Applying the boundary condition (ii) one gets:

$$k_n(z)\frac{\partial}{\partial z}c_{y_n}(l,z)=0 \quad \text{at} \quad z=h \quad \text{then :}$$

$$\frac{\partial}{\partial z}c_{y_n}(l,z)=R_nc_{y_n}(l,0)\left(\sinh R_nz+\frac{2v_d}{R_a}\cosh R_nz\right)-\frac{Q}{R_a}R_n\cosh R_n(z-h_s)H(z-h_s)-\frac{Q}{R_a}\sinh R_n(zh_s)\frac{\partial}{\partial z}H(z-h_s) \tag{40}$$

$$c_{y_n}(l,0)\left(\sinh R_nz+\frac{2v_d}{R_a}\cosh R_nz\right)=\frac{Q}{R_a}\cosh\left[R_n(h-h_s)H(h-h_s)\right] \tag{41}$$

$$c_{y_n}(l,0)=\frac{Q}{R_a}\frac{\cosh R_n(h-h_s)}{\left(\sinh R_nh+\frac{2v_d}{R_a}\cosh R_nh\right)} \tag{42}$$

Substituting equation (42) for equation (40) then one gets:

$$\tilde{c}_{y_n}(l,z)=\frac{Q}{\sqrt{lk_n}}\frac{\cosh\sqrt{\frac{l}{k_n}}(h-h_s)\left(\cosh R_nz+\frac{2v_d}{k_n}\sinh R_nz\right)}{\left(\sinh R_nh+\frac{2v_d}{R_a}\cosh R_nh\right)}-\frac{Q}{R_a}\sinh R_n(z-h_s)*H(z-h_s) \tag{43}$$

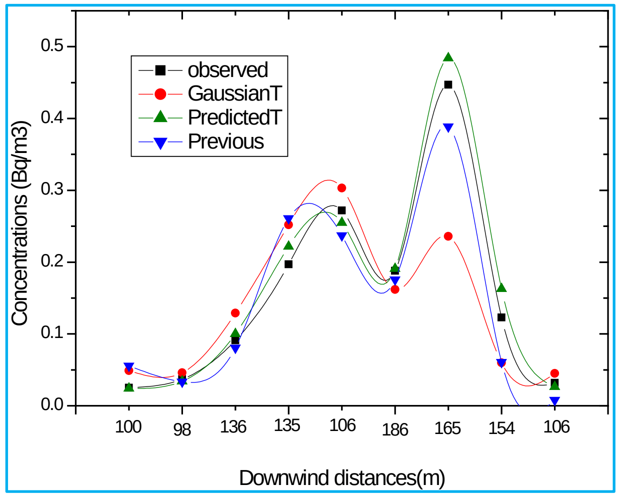

Fig. 3. Shows that the relation between the Gaussian, previous work, predicted model and observed Concentrations (Bq/m^3) *via* downwind distances

TABLE 5

Observed, the Gaussian, Previous work and predicted concentrations for Run 9 experiments

Test	Downwind distance (m)	Observed conc.(Bq/m^3)	Gaussian conc.Eqns. (28, c, d, e) (Bq/m^3)	Previous workEssa *et al.*, 2019(Bq/m3)	Predicted conc. Eq.(44) (Bq/m^3)
1	100	0.025	0.049	0.0558	0.02408
2	98	0.037	0.046	0.0334	0.034
3	136	0.091	0.129	0.0809	0.1
4	135	0.197	0.252	0.2609	0.2217
5	106	0.272	0.303	0.2373	0.255
6	186	0.188	0.162	0.1757	0.1907
7	165	0.447	0.236	0.3888	0.484
8	154	0.123	0.060	0.0609	0.163
9	106	0.032	0.045	0.0082	0.0265

By using Gaussian quadrature formulas, one gets:

$$c_{y_n}(y,z,t)=\frac{Q\exp\left(\frac{-y^2}{2\sigma_y}-\nu t\right)}{\sqrt{2\pi}\sigma_y}\sum_{i=1}^{8}a_i\left(\frac{p_i}{t}\right)\frac{1}{\sqrt{\frac{k_n(z)p_i}{t}}}\frac{\cosh\sqrt{\frac{p_i}{tk_n}}(h-h_s)\left(\cosh\sqrt{\frac{p_i}{tk_n}}z+\frac{2v_d}{k_n}\sinh\sqrt{\frac{p_i}{tk_n}}z\right)}{\left(\sinh\sqrt{\frac{p_i}{tk_n}}h+\frac{2v_d}{R_a}\cosh\sqrt{\frac{p_i}{tk_n}}h\right)} \tag{44}$$

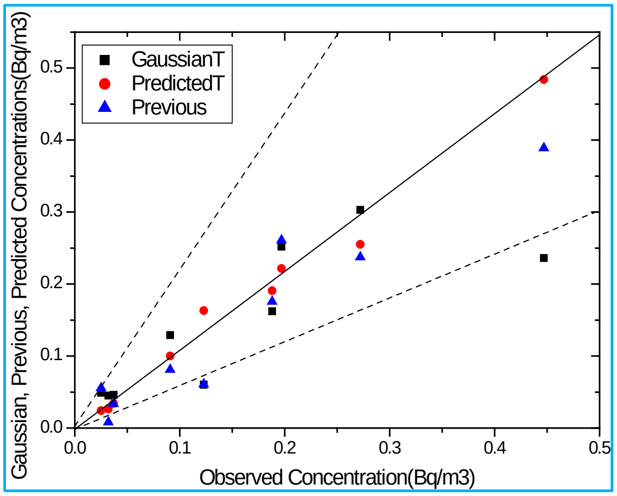

Fig. 4. Shows that the achieving between the Gaussian, previous work, predicted concentrations with observed concentration

TABLE 6

Comparison between the Gaussian, predicted and observed concentrations in unstable conditions

	NMSE	FB	COR	FAC2
Gaussian Equations (28, c, d, e)	0.28	0.096	0.81	0.91
Pred. Eqn. (44)	0.02	-0.06	0.99	1.1
Previous work	0.05	0.08	0.96	0.92

The observed concentration of I^{135}, the predicted concentrations of Eqns. (28, c, d, e), previous work (Essa *et al.*, 2019) and Eqn. (44) below the plume 's centerline are also presented in Table 5. A comparison between the Gaussian, previous work, predicted and observed concentrations of radioactive I^{135} *via* downwind distance in unstable conditions at Inshas are shown in Fig. 3, also, the relation between the Gaussian, previous work, predicted and observed concentration data are shown in Fig. 4.

One finds that the best model when we use Eqn. (44) and the previous work because of the vertical eddy diffusivity as a function of the power law of the vertical height also, the predicted, previous work (Essa *et al.*, 2019) and the Gaussian plume model of Eqns. (28, c, d, e)lie inside a factor of two. The predicted, the previous work and the Gaussian plume model achieved 100%, 0.92% and 91% respectively.

3.1. *Statistical Technique of instanteous source*

Comparing between the Gaussian, predicted and observed concentrations was introduced by (Hanna, 1989).

where, NMSE is the normalized mean square error, FB is the fraction bias, COR is the correlation coefficient and FAC2 is a factor of two.

One can easily see from Table 6,the statistical technique shows that the entire proposed models and the previous work are located inside a factor of two with observed concentration data. Also, the statistical shows that the predicted model Eqn. (44) and the previous work are the best for NMSE, FB, COR and FAC2 than the Gaussian model Eqns. (28, c, d, e) which acts a good model.

4. Conclusions

In this work, one-dimension time-dependent and three dimensions of the advection-diffusion equation (ADE) of the steady state has been solved analytically. To get the concentration into the atmospheric boundary layer (ABL) taking into account the assumption that the ABL height (h) and the wind speed are divided into sub-layers intervals within each rectangular area the ADE is estimated by using the Laplace transform method assuming the mean values of wind speed and eddy diffusivity. The proposed model, the Gaussian plume model and previous work (Essa *et al.*, 2019) was compared with the observed concentration of Iodine-135 which was measured at the Egyptian Atomic Energy Authority, Nuclear Research Reactor, Inshas, Cairo, Egypt.

From the three dimensions of the advection-diffusion equation (ADE) of the steady state in continuous source, we get the best model when uses Eqn. (27) than the Gaussian plume model, because of the vertical eddy diffusivity as a function of the power law into the vertical height. The predicted models achieved 99%. Also, the Gaussian model using Eqns. (6, 10, 11) gives a good result and achieved 79% because of the strongest of the dispersion parameters (10, 11).

Also, from one-dimension time-dependent the advection-diffusion equation in instantaneous source, one finds that the best model when we, use Eqn. (44) and the previous work (Essa *et al.*, 2019) because of the vertical eddy diffusivity as a function of the power law into the vertical height, also, the predicted, previous models and the Gaussian lie inside a factor of two. The predicted model, the previous work and the Gaussian model achieved 100%, 92% and 91% respectively.

Also, the statistical analysis shows that there is a good agreement between the proposed, the previous work (Essa *et al.*, 2019) and experimental values of concentration than the Gaussian plume model.

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