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Asymptotic solution for 3D Lee waves across Assam-Burma hills

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सार – इस शोध पत्र में असम बर्मा की पहाड़ियों (एबीएच) के ऊपर से बहने वाली ली वेब के उर्ध्ववाह/अद्योप्रवाह के मौलिक बहाव को आदर्श रूप में प्रस्तुत करने के लिए एक उपगामी निष्कर्ष प्राप्त करने का प्रयास किया गया है जहाँ स्थयित्व और पवन का मौलिक बहाव ऊँचाई के साथ निश्चर रहते हैं। असम–बर्मा पहाड़ियों के सन्निकट दो त्रिआयामी दीर्घवृन्ताकार रोधिकाओं के द्वारा पहँचा गया है जो कुछ निश्चित दूरी पर एक घाटी द्वारा विलग है। इसको सहजता से समझने के लिए पवन के मौलिक बहाव को रोधिकाओं के बड़े–बड़े किनारों के लंबवत एक अवयव माना गया है। इसके लिए गवर्निंग समीकरण में क्षोभ तकनीक को लागू किया गया है। क्षोभ उदग्र वेग (w') और वायु प्रवाही रेखा की दूरी (η') को द्विशः समाकलों में दर्शाया गया है जिसे उपगामी विस्तार के रूप में समझने का प्रयास किया गया है। इससे प्राप्त परिणाम सुसंगत पाए गए है और इसकी तुलना पूर्ववर्ती अन्वेषण कर्त्ताओं द्वारा प्राप्त किए गए परिणामों से की गई है।

ABSTRACT. An attempt has been made to obtain an asymptotic solution for the updraft/downdraft associated with Lee wave across the Assam-Burma hills (ABH) for an idealized basic flow, where, both stability and wind in the basic flow remain invariant with height. ABH has been approximated by two three dimensional elliptical barriers, separated by a valley of some finite distance. For simplicity, the basic flow has been assumed to have only one component normal to the major ridges of the barriers. For this purpose, perturbation technique has been applied to the governing equations. The perturbation vertical velocity (w') and streamline displacement (η') are expressed as double integrals, which have been attempted to approximate as an asymptotic expansion. The results obtained are found to be consistent and have been compared with the results obtained by earlier investigators.

Key word – Assam Burma hills, Asymptotic solution, 3D Lee wave.

1. Introduction

The problem of airflow over an orographic barrier and in particular, the formation of stationary atmospheric Lee waves has received considerable theoretical attention in the past. In a stably stratified atmosphere a fluid parcel, displaced vertically, undergoes buoyancy oscillation which gives rise to gravity waves. Now, these gravity waves can propagate vertically to a great distance carrying energy and momentum to higher levels in the atmosphere. Sometimes, they are associated with the formation of Clear Air Turbulence (CAT). The information about standing waves, which under favorable meteorological conditions form on the Lee side of the mountain barrier, is very important for the safety of aviation. Many aircraft accidents reported in mountainous areas are often attributed to the vertical velocities of large magnitude associated with the Lee waves.

Studies on the perturbation to a stably stratified air stream by an obstacle can be broadly divided into two categories. In one category the obstacle is assumed to have an infinite extension in the crosswind direction, so that the flow is essentially two-dimensional (2-D). In the other category the obstacle is assumed to have a finite extension in the crosswind direction and consequently the flow is three-dimensional (3-D). One fundamental difference between the 2-D and 3-D approach is the direction of propagation of wave energy away from the mountain. In two dimensions as the mountain becomes wider and the flow more nearly hydrostatic, the group velocity (relative to the mountain) becomes directed vertically with the result that the wave energy is found directly above the mountain. This result does not carry over to three dimensions. Some of the hydrostatically waves generated by 3-D mountain lie down stream of the mountain and to the side tending to form trailing wedges of vertical motion. Thus for practical and theoretical reasons, it is necessary to understand the three-dimensional mountain flow problem.

Wurtele (1957) represented the 3-D orographic barrier in the form of semi-infinite plateau of height 'h' with narrow width '2b' in the crosswind direction. He considered the incoming wind (U) and buoyancy frequency (N) to be independent of height. His theory predicted the region of updraft, which had a horseshoe shape, and was located some distance downstream of the barrier. Crapper (1959) presented a 3-D small perturbation approach of waves produced in a stably stratified air stream flowing over a mountain. He obtained the fundamental solution for a doublet disturbance in an air stream in which Scorers parameter remains constant and then it was extended to that for a disturbance caused by a circular mountain in the same air-stream. He showed that circular mountain can give rise to waves which have greater amplitude than those produced by an infinite ridge in the same air-stream. Crapper (1962) considered the airflow across a 3-D barrier with elliptical contour for two types of air-stream. In one case the Scorer parameter l was constant with height, in other case it was assumed to fall off exponentially with height. In each of the above cases $\frac{1}{U}\frac{d^2U}{dz^2} = q^2$ was kept constant. The result showed that when l is constant, then the form of the waves was determined by the value of q. He also showed that when l falls off exponentially, the waves closely resembled ship waves for any value of q.

Sawyar (1962) studied gravity waves in the atmosphere as a 3-D problem. He derived an equation, for the vertical variation of the amplitude of the standing waves, when the wind varied with height and the wave was periodic in the horizontal. He solved the equation numerically for specified two or three layer atmosphere to determine possible wavelengths in the horizontal directions for Lee waves. He obtained results for the cases when wind direction changed with height as well as for the cases when wind direction remained same in the vertical. He showed interestingly that Scorer's (1949) condition for the occurrence of Lee wave was no longer applicable for wave motion in 3-D. He showed that in 3-D Lee waves are always possible in a two-layer atmosphere. Onishi (1969) solved 3-D linearized equations for arbitrary upstream conditions by including friction in the governing equations. Pekelis (1971) extended his 2-D work to solve linearized 3-D problem. Vertical velocity fields obtained by him compare well with those of Sawyar (1962). Smith (1980) examined the stratified hydrostatic flow over a bell shaped 3-D isolated mountain using linear theory. Solutions for various parts of the flow field were obtained using analytical method and numerical Fourier analysis. The flow aloft was found to be composed of vertically propagating mountain waves. The maximum amplitude of these waves occurred directly over the mountain, but there was considerable wave energy, trailing downstream along the parabolas $y^2 = \frac{Nzax}{U}$; where U, N are respectively the constant basic zonal wind and buoyancy frequency. Bluemen and Dietze (1981) considered a 3-D linear hydrostatic model of stationary mountain wave in a stably stratified air-stream. They took both the incoming flow and Burnt-Vaisalla frequency to be independent of height, but lateral variation of incoming flow was incorporated by assuming a hyperbolic secant profile ($U = \text{Sechy}$). The results in their solution for different shape of hill showed that the pressure pattern and the velocity at the ground level were similar in many respects to the field obtained by Smith (1980) for constant basic flow. The incoming air-stream tends to circumvent the hill resulting in a permanent streamline deflection. Somieski (1981) studied the stratified hydrostatic flow over a three dimensional circular mountain. He derived a 2nd order wave equation from the primitive equation including constant rotation and vertical wind shear of the mean flow. He solved the equation numerically. He showed that in case of no shear and constant static stability, the nodal lines are parabolic for a circular mountain of diameter 50 km. Bluemen and Dietze (1982) extended their earlier model by including the vertical variation of the basic flow and static stability. To take into account the vertical structure of basic state, they introduced stretched vertical co-ordinate. The energy flux computed by them was compared with the results of Elliassen and Palm (1961). Olafssen and Bougeault (1996) explored the hydrostatic flow over an elliptical mountain barrier of aspect ratio 5. They took upstream profiles of wind (U), stability (N) constant and ignored the effect of Coriolis force. Under such conditions their result showed the flow characteristics to be dependent mainly on the non-dimensional mountain height $\frac{Nh}{U}$. They found that for all values of $\frac{Nh}{U}$, a substantial part of the flow was diverted vertically above the mountain. They found generation of potential vorticity in the wake of the mountain, leading to the creation of Lee vortices.

In India the problem of mountain waves has been studied by Das (1964), Sarker (1965, 1966, 1967), De (1971), Sinha Ray (1988), Tyagi & Madan (1989) and Kumar *et al*. (1995), Dutta (2003, 2005, 2007a, 2007b), Dutta *et al*., (2002), Dutta *et al*. (2006), Dutta & Naresh Kumar (2005), etc. Among all, Das (1964) first addressed the issue from a 3-D aspects. He studied the influence of the Himalayas, approximated as a large 3-D circular mountain, using a linear baroclinic model which included the variation of '*f*' with latitude. The solutions were obtained for the downstream waves in an asymptotic form. It was shown that the wave distortion depends on a non-dimensional parameter, which is a combination of Froude number $\left(U^2/gd\right)$, Rossby number $\left(U/fL\right)$ and the static stability $\left(l = \frac{1}{\theta}\frac{d\theta}{dz}\right)$. Considering the center of the circular mountain at 85° E and 30°-40° N, the major trough line was found between 105°-110° E, which is in reasonable agreement with the observations over eastern Tibet in winter. Afterwards 3-D study on mountain wave and its different aspects were studied by Dutta *et al*., (2002) and subsequently by Dutta (2003), Dutta *et al*. (2006) and Dutta (2005, 2007a, 2007b). These studies addressed the Lee waves across the Western Ghats and the Khasi-Jayantia hills only. In India problems on Lee waves across the Assam-Burma hills was first addressed by De (1970) and subsequently by De (1971), Farooqui and De (1974), Dutta and Naresh Kumar (2005) etc. De (1970,1971) computed the wavelength of the Lee waves over the Assam-Burma hills using an approach, similar to Sarker (1966, 1967) with necessary modification for the mountain profile and wind direction in that region. Computed wavelength varies between 17 and 34 km and agreed well with those observed from satellite pictures. Farooqui and De (1974) used a two dimensional model to calculate the flow over a small obstacle (half width 2 km), a large obstacle (half width of 20 km) and across the Assam hills (200-300 km). Their results in the later experiment show long waves of length (20-40 km) and other large perturbations mainly between heights 1 and 9 km. From 9 to 15 km perturbations are very small. However, these studies on Lee waves across the Assam-Burma hills are of 2-D.

This study aims at developing a 3-D mesoscale Lee wave model for the Assam-Burma hills (ABH) and at obtaining an analytical solution for the updraft and vertical displacement associated with 3-D mesoscale Lee waves across ABH, following an asymptotic approximation.

2. Data

Input data for the proposed model consists of geopotential height, pressure, dry bulb temperature and horizontal components of wind at different levels at a station far upstream of the barrier. As the ABH is north south oriented, hence when it is approximated analytically by two 3-D elliptical barriers, separated by a valley, the major ridges become north south oriented. Hence the zonal component of the basic flow only interacts effectively with the ABH to give rise to Lee waves. The only station on the upstream side is Guwahati (26.19° N Latitude and 91.73° E Longitude). Hence the RS/RW data of Guwahati for those dates, which corresponds to the observed Lee waves across ABH, as reported by De (1970, 1971) and Farooqui and De (1974), has been obtained from Archive of India Meteorological Department, Pune.

3. Methodology

Similar to Dutta *et al*. (2002), in the present study also, an adiabatic, steady state, non-rotating and laminar flow of a vertically unbounded stratified and Bossiness fluid across a 3D mesoscale orographic barrier, has been considered. Present study is similar to the study of Dutta *et al.* (2002) in most of the aspects, except the lower boundary condition. In Dutta *et al.* (2002) lower boundary was a 3-D elliptical barrier whereas in the present study the barrier is approximated analytically by two 3-D elliptical barriers, separated by a valley, Similar to Dutta *et al*. (2002), in the present study also it is assumed that the basic flow (U) is normal to the major ridges of elliptical barriers and it is constant with height and the buoyancy frequency (N) is also assumed to be constant with height and a rectangular co-ordinate system in which, x axis points towards east, y axis towards north and z axis vertically upward is considered. Using the technique followed by Dutta *et al*. (2002), we obtain the following vertical structure equations for perturbation vertical velocity (w') and for perturbation vertical streamline displacement (η') :

$$\frac{\partial^2 \hat{w}_1}{\partial z^2} + \left(m^2 - \frac{1}{2\rho_0}\frac{d^2\rho_0}{dz^2} + \frac{1}{4{\rho_0}^2}\left(\frac{d\rho_0}{dz}\right)^2 \right)\hat{w}_1 = 0 \quad (1)$$

$$\frac{\partial^2 \hat{\eta}_1}{\partial z^2} + \left(m^2 - \frac{1}{2\rho_0}\frac{d^2\rho_0}{dz^2} + \frac{1}{4{\rho_0}^2}\left(\frac{d\rho_0}{dz}\right)^2 \right)\hat{\eta}_1 = 0 \quad (2)$$

Where $\hat{w}, \hat{w}_1, \hat{\eta}, \hat{\eta}_1$ are double Fourier transforms of w', w_1', η', η_1' respectively,

$$w'(x,y,z) = w_1(x,y,z)c(z) \quad (3)$$

$$\eta'(x,y,z) = \eta_1(x,y,z)c(z) \quad (4)$$

$$m^2 = \frac{k^2 + l^2}{k^2}\left(\frac{N^2}{U^2} - k^2\right) \quad (5)$$

Where

$$c(z)=\sqrt{\frac{\rho_0(0)}{\rho_0(z)}} \tag{6}$$

and $\rho_0(z)$ is the basic state density at the level z.

Since the last two terms in the brackets in equations (1) & (2) are very small in magnitude, hence they can be neglected, so that these equations are simplified and written as

$$\frac{\partial^2 \hat{w}_1}{\partial z^2}+m^2\hat{w}_1=0 \tag{7}$$

and

$$\frac{\partial^2 \hat{\eta}_1}{\partial z^2}+m^2\hat{\eta}_1=0 \tag{8}$$

respectively.

For vertically propagating wave, the solutions of (7) and (8) can be taken as

$$\hat{w}_1(k,l,z)=\hat{w}_1(k,l,0)e^{imz} \tag{9}$$

and

$$\hat{\eta}_1(k,l,z)=\hat{\eta}_1(k,l,0)e^{imz} \tag{10}$$

respectively.

Now at the lower boundary *i.e.*, at surface, the airflow follows the contour of the mountain, the profile of which is given by

$$h(x,y)=\frac{h_1}{1+\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}}+\frac{h_2}{1+\dfrac{(x-d)^2}{a^2}+\dfrac{y^2}{b^2}} \tag{11}$$

Where a = 20 km, b = 50 km, d = 55 km, h_1 = 0.9km, h_2 = 0.7 km.

Now the double Fourier transform of (11) is given by

$$\hat{h}(k,l)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(x,y)e^{-i(kx+ly)}\mathrm{d}x\mathrm{d}y$$

$$=2\pi ab\left(h_1+h_2e^{-ikd}\right)K_0\left(\sqrt{a^2k^2+b^2l^2}\right) \tag{12}$$

where $K_0\left(\sqrt{a^2k^2+b^2l^2}\right)$ is the Bessel function of second kind of order zero. Details of the derivation are given in Appendix I.

Thus at the ground surface, $\eta_1(x,y,0)=\eta(x,y,0)=h(x,y)$

Hence,

$$\hat{n}_1(k,l,0)=\hat{h}(k,l)=2\pi ab\left(h_1+h_2e^{-ikd}\right)K_0\left(\sqrt{a^2k^2+b^2l^2}\right) \tag{13}$$

Now the linearized lower boundary condition for w_1' is given by

$$\begin{aligned} w_1'(x,y,0)=w'(x,y,0)&=U\,\frac{\partial\eta'(x,y,0)}{\partial x}\\ &=U\,\frac{\partial\eta_1'(x,y,0)}{\partial x}\end{aligned} \tag{14}$$

Hence,

$$\begin{aligned}\hat{w}_1(k,l,0)=ikU\hat{\eta}_1(k,l,0)&=2\pi\, i\, kUab\\ &(h_1+h_2e^{-ikd})K_0\left(\sqrt{a^2k^2+b^2l^2}\right)\end{aligned} \tag{15}$$

Therefore,

$$\hat{\eta}_1(k,l,z)=2\pi ab(h_1+h_2e^{-ikd})K_0\left(\sqrt{a^2k^2+b^2l^2}\right)e^{imz} \tag{16}$$

and,

$$\hat{w}_1(k,l,z)=2\pi ikUab(h_1+h_2e^{-ikd})K_0\left(\sqrt{a^2k^2+b^2l^2}\right)e^{imz} \tag{17}$$

Hence, streamline displacement and perturbation vertical velocity at any point (x,y,z) are given by

$$\eta_1'(x,y,z)=\mathrm{Re}\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{\eta}_1(k,l,z)e^{i(kx+ly)}\mathrm{d}k\mathrm{d}l$$

$$= \frac{ab}{2\pi} \mathrm{Re} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (h_1 + h_2 e^{-ikd}) K_0(\sqrt{a^2k^2 + b^2l^2}) e^{i(kx+ly+mz)} \mathrm{d}k\, \mathrm{d}l \quad (18)$$

$$w_1'(x,y,z) = \mathrm{Re} \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{w}_1(k,l,z) e^{i(kx+ly)} \mathrm{d}k \mathrm{d}l$$

$$= \frac{Uab}{2\pi} \mathrm{Re} \int\int ik(h_1 + h_2 e^{-ikd}) K_0(\sqrt{a^2k^2 + b^2l^2}) e^{i(kx+ly+mz)} \mathrm{d}k \mathrm{d}l \quad (19)$$

If $\frac{U}{N}$ is taken as the unit of length, then the integrands in Eqns. (18) and (19) may be rendered non-dimensional by the following substitution:

$$X = \frac{xN}{U},\ Y = \frac{yN}{U}, Z = \frac{zN}{U} \text{ and}$$

$$\lambda = k\frac{U}{N}, \mu = l\frac{U}{N}, \nu = m\frac{U}{N}.$$

With the above substitution Eqns. (18) and (19) reduces to $\frac{abN^2}{2\pi U^2} \mathrm{Re} I_1$ and $\frac{abUN^3}{2\pi U^3} \mathrm{Re} I_2$ respectively.

Where

$$I_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(h_1 + h_2 e^{-\frac{iN\lambda d}{U}} \right) K_0\left(\frac{N}{U} \sqrt{a^2\lambda^2 + b^2\mu^2} \right) e^{i(\lambda X + \mu Y + \nu Z)} \mathrm{d}\lambda \mathrm{d}\mu \quad (20)$$

and

$$I_2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} i\lambda \left(h_1 + h_2 e^{\frac{-iN\lambda d}{U}} \right) K_0\left(\frac{N}{U} \sqrt{a^2\lambda^2 + b^2\mu^2} \right) e^{i(\lambda X + \mu Y + \nu Z)} \mathrm{d}\lambda \mathrm{d}\mu \quad (21)$$

The double integrals I_1 and I_2 are difficult to evaluate analytically. So they are responsible to the method of stationary phase. According to this method, first those points in the wave number (k,l) domain are found out, where the phase $(kx + ly + mz)$ is stationary. Those points are termed as saddle points. Then the entire integrand is expanded in Taylor's series about the saddle point and the first term of the expansion is retained as the asymptotic approximation of the integrals, which is valid at far down wind location of the mountain. The asymptotic expansion for η' is given by

$$\eta'(x,y,z) = c(z)\eta_1'(x,y,z) = \exp\left(\frac{g - R^\bullet\gamma}{2R^\bullet\overline{T}} z \right) \eta_1'(x,y,z) \quad (22)$$

Where,

$$\eta_1'(x,y,z) = \frac{abN^2}{2U^2} \frac{XZ\sqrt{\rho^4 + X^2Y^2}\left[h_1 \cos\left(\frac{RZ}{\rho} \right) + h_2 \cos\left(\frac{RZ}{\rho} - \frac{NXZd}{UR\rho} \right) \right] K_0\left(\frac{XZN\sqrt{a^2\rho^4 + b^2X^2Y^2}}{RU\rho^3} \right)}{R^2\rho^3 \sqrt{\left\{ 1 + 4\left(\frac{XYZR}{\rho^4 + X^2Y^2} \right)^2 \right\}}}$$

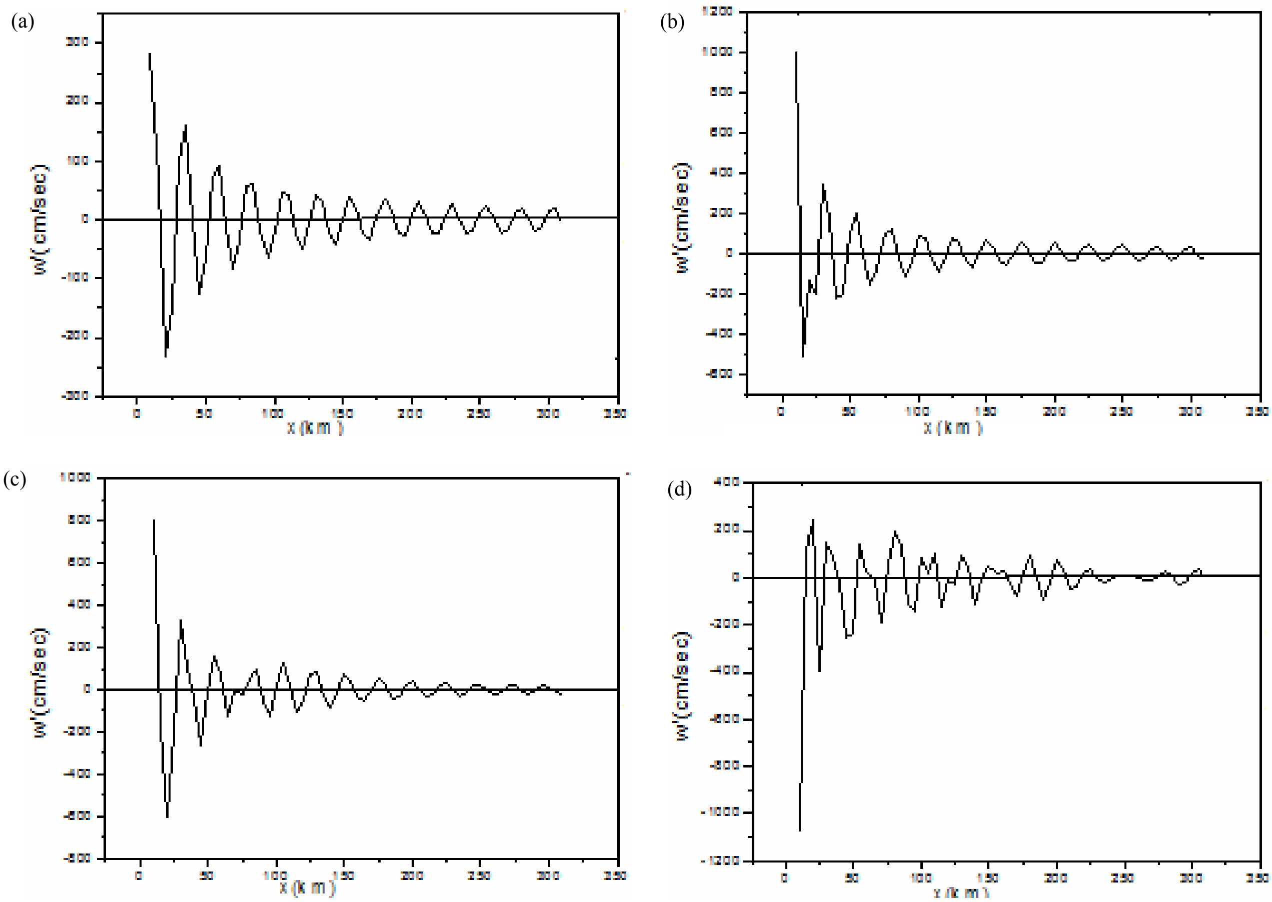
Details of the above derivation are given in Appendix 2.

Where, $c(z) \approx \exp\left(\frac{g - R^\bullet\gamma}{2R^\bullet\overline{T}} z \right)$ [taken from Dutta *et al.*, (2002)]

$R^\bullet$ is the specific gas constant of the atmosphere and γ is the lapse rate of the basic state and $R^2 = X^2 + Y^2 + Z^2, \rho^2 = Y^2 + Z^2$.

Similarly, the asymptotic expansion for perturbation vertical velocity w' is given by

$$w'(x,y,z) = \exp\left(\frac{g - R^\bullet\gamma}{2R^\bullet\overline{T}} z \right) w_1'(x,y,z) \quad (23)$$



Figs. 1(a-d). Downstream variation of w′ in the central plane at (a) 1.5 km level for ABH, (b) 3 km level for ABH, (c) 6 km level for ABH and (d) 10 km level for ABH

where,

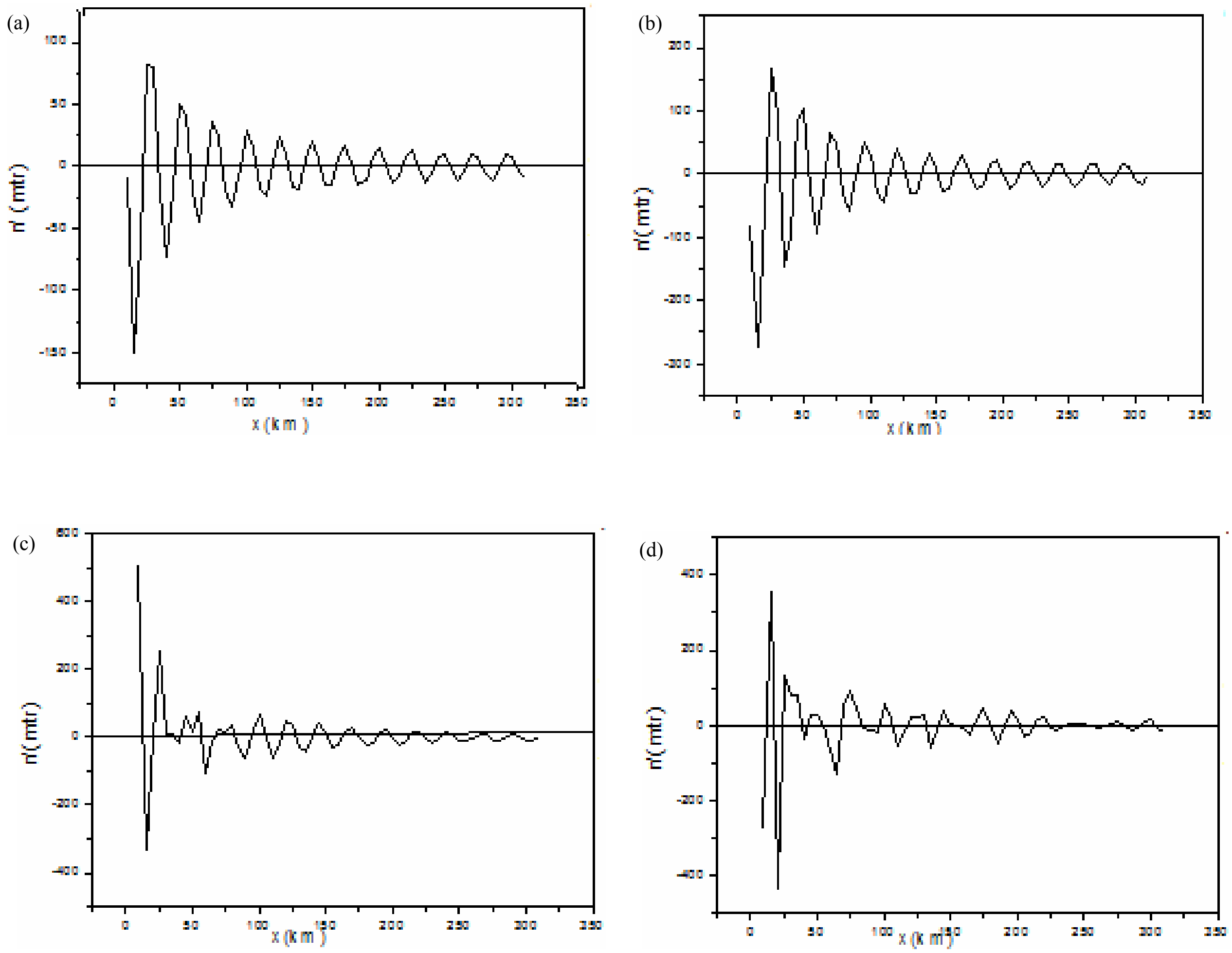
$$w_1'(x,y,z) = -\frac{abN^3}{U^2}\,\frac{X^2Z^2\sqrt{\left(\rho^4+X^2Y^2\right)}\left[h_1\sin\left(\frac{RZ}{\rho}\right)+h_2\sin\left(\frac{RZ}{\rho}-\frac{NXZd}{UR\rho}\right)\right]K_0\left(\frac{XZN\sqrt{a^2\rho^4+b^2X^2Y^2}}{RU\rho^3}\right)}{2R^3\rho^4\sqrt{\left\{1+4\left(\frac{XYZR}{\rho^4+X^2Y^2}\right)^2\right\}}} \tag{24}$$

Details of the above derivation are given in Appendix 2.

4. Results and discussions

Now from (23) it is clear that at $z = 0$, *i.e.*, at the surface w' vanishes except at $x = 0$, $y = 0$. This should be so for Lee wave. The geometrical description of wave pattern in 3-D is obtained by substituting $y = 0$. At $y = 0$, w' is given by:

$$w'(x,0,z) = -\exp\left(\frac{g-R^{\bullet}\gamma}{2R^{\bullet}\bar{T}}z\right)\frac{abN^2}{2U}\frac{x^2}{\left(\sqrt{x^2+z^2}\right)^3}K_0\left(\frac{ax\mathrm{N}}{U\sqrt{x^2+z^2}}\right)$$
$$\left[h_1\sin\left(\frac{\mathrm{N}}{U}\sqrt{x^2+z^2}\right)+h_2\sin\left(\frac{\mathrm{N}}{U}\sqrt{x^2+z^2}-\frac{\mathrm{N}}{U}\frac{xd}{\sqrt{x^2+z^2}}\right)\right] \tag{25}$$



Figs. 2(a-d). Downstream variation of n′ in the central plane at (a) 1.5 km level for ABH, (b) 3 km level for ABH, (c) 6 km level for ABH and (d) 10 km level for ABH

From this expression it follows that at any level, perturbation vertical velocity decays downstream of the barrier in the central plane, This may be attributed to the presence of the Bessel function and the term $\frac{x^2}{\left(x^2+z^2\right)^{\frac{3}{2}}}$. Figs. 1(a-d) show the downstream variation of w' in the central plane at 1.5 km, 3.0 km, 6.0 km and at 10.0 km above mean sea level, which approximately resemble to 850 hPa, 700 hPa, 500 hPa and 300 hPa respectively. Each of them shows downstream decay in the amplitude of w' in the central plane.

Similarly at $y = 0$, η' is given by from (22)

$$\eta'(x,0,z) = \exp\left(\frac{g-R^{\bullet}\gamma}{2R^{\bullet}\overline{T}}z\right)\frac{ab\mathrm{N}}{2U}\frac{x}{\left(x^2+z^2\right)}K_0\left(\frac{ax\mathrm{N}}{U\sqrt{x^2+z^2}}\right)\left[h_1\cos\left(\frac{\mathrm{N}}{U}\sqrt{x^2+z^2}\right)+h_2\cos\left(\frac{\mathrm{N}}{U}\sqrt{x^2+z^2}-\frac{\mathrm{N}}{U}\frac{xd}{\sqrt{x^2+z^2}}\right)\right] \tag{26}$$

which appears to decay downstream of the barrier in the central plane due presence of the Bessel function and the term $\frac{x}{\left(x^2+z^2\right)}$. Figs. 2(a-d) show the downstream variation of η' in the central plane at 1.5 km, 3.0 km, 6.0 km and at 10.0 km above mean sea level, which approximately resemble to 850 hPa, 700 hPa, 500 hPa and 300 hPa respectively. Each of them shows downstream decay in the amplitude of η' in the central plane.

The nodal lines (on any horizontal plan $z = z_0$) of the respective field at any level corresponds to maximum or minimum values of the field. This corresponds to $\frac{Rz_0}{\rho} = n\pi \; or \; \frac{(2n\pm 1)\pi}{2}$ and $\frac{RZ_0}{\rho} - \frac{NX\mathrm{d}Z_0}{UR\rho} = m\pi$ or $(2m\pm 1)\frac{\pi}{2}$, where m, n are integers.

Dutta *et al.*, (2002) have shown that, $\frac{Rz_0}{\rho} = n\pi \; or \; \frac{(2n\pm 1)\pi}{2}$ corresponds to a family of hyperbolas with length of transversal and conjugate axes respectively $2\sqrt{n^2\pi^2 - Z_0{}^2}$ and $2Z_0$. The latus rectum of the hyperbola on the horizontal plane $z = z_0$ is given by $l(z_0) = \frac{2z_0^2}{\sqrt{n^2\pi^2 - z_0^2}}$. Clearly as the value of z_0 increases, $l(z_0) = \frac{2z_0^2}{\sqrt{n^2\pi^2 - z_0^2}}$ also increases, as a result of which nodal lines spread laterally with height.

Again another nodal lines of the respective field at any level corresponds to maximum or minimum values of the field. This corresponds to $\frac{RZ_0}{\rho} - \frac{NX\mathrm{d}Z_0}{UR\rho} = m\pi$ or $(2m\pm 1)\frac{\pi}{2}$.

Let $\frac{N}{U} = S_1$ where $R^2 = X^2 + Y^2 + Z_0^2$ and $\rho^2 = Y^2 + Z_0^2$

or $$\frac{RZ_0}{\rho} - \frac{XS_1Z_0d}{R\rho} = m\pi$$

or $$\frac{Z_0}{\rho}\left(R - \frac{XS_1d}{R}\right) = m\pi$$

or $$Z_0\left(R - \frac{XS_1d}{R}\right) = m\pi\rho$$

or $$Z_0\left(R^2 - XS_1d\right) = m\pi\rho R$$

or $$Z_0^2\left(X^2 + Y^2 + Z_0^2 - XS_1d\right)^2 = m^2\pi^2\rho^2R^2$$

or $$Z_0^2\left(X^2 + Y^2 + Z_0^2 - XS_1d\right)^2 = m^2\pi^2\left(Y^2 + Z_0^2\right)\left(X^2 + Y^2 + Z_0^2\right)$$

or $$Z_0^2\left(X^2 + Y^2 + Z_0^2\right)^2 - 2X\left(X^2 + Y^2 + Z_0^2\right)Z_0^2S_1d + X^2S_1^2Z_0^2d^2 = m^2\pi^2\left(Y^2 + Z_0^2\right)\left(X^2 + Y^2 + Z_0^2\right)$$

or $$\left(X^2 + Y^2 + Z_0^2\right)\left[Z_0^2\left(X^2 + Y^2 + Z_0^2\right) - m^2\pi^2\left(Y^2 + Z_0^2\right) - 2XZ_0^2S_1d\right] + X^2S_1^2Z_0^2d^2 = 0 \tag{27}$$

Now for the ABH $S_1 = \frac{N}{U}$ where $N = 10^{-2}s^{-1} = 0.01\mathrm{s}^{-1}$ and $U = 10\mathrm{m/s}$.

Therefore

$$S_1^2 = \frac{N^2}{U^2} = \frac{(0.01)^2}{10^2} = 0.000001 \approx 0$$

So, we neglect the last term from (27) and we get

$$\left(X^2 + Y^2 + Z_0^2\right)\left[Z_0^2\left(X^2 + Y^2 + Z_0^2\right) - m^2\pi^2\left(Y^2 + Z_0^2\right) - 2XZ_0^2S_1d\right] = 0$$

$$\therefore \left(X^2 + Y^2 + Z_0^2\right)\left[Z_0^2X^2 + \left(Z_0^2 - m^2\pi^2\right)Y^2 - 2XZ_0^2S_1d + Z_0^2\left(Z_0^2 - m^2\pi^2\right)\right] = 0 \tag{28}$$

Where $m = 0, \pm 1, \pm 2, \pm 3, \ldots\ldots$

From Eqn. (28) we get,

or, $X^2 + Y^2 + Z_0^2 = 0$, it represents imaginary circle, So we neglect it.

Either,

$$Z_0^2 X^2 + \left(Z_0^2 - m^2\pi^2\right)Y^2 - 2XZ_0^2 S_1 d + Z_0^2\left(Z_0^2 - m^2\pi^2\right) = 0 \tag{29}$$

Case 1. If $m = 0$ then the equation (29) represents the circle.

Therefore, $\frac{RZ_0}{\rho} - \frac{NX\,\mathrm{d}Z_0}{UR\rho} = m\pi$ or $(2m \pm 1)\frac{\pi}{2}$ it shown that, corresponds to a family of circles.

Case 2. If $m \neq 0$ then we have following

We compare the equation (29) with the following equation

$AX^2 + 2HXY + BY^2 + 2GX + 2FY + C = 0$

we get

$$A = Z_0^2, B = Z_0^2 - m^2\pi^2, H = 0 \text{ now}$$
$$H^2 - AB = -Z_0^2\left(Z_0^2 - m^2\pi^2\right) \tag{30}$$

(*i*) if $Z_0^2 - m^2\pi^2 < 0$ then $H^2 - AB > 0$ implies hyperbola.

Therefore , eqn. (29) represents a hyperbola.

Hence, $\frac{RZ_0}{\rho} - \frac{NX\,\mathrm{d}Z_0}{UR\rho} = m\pi$ or $(2m \pm 1)\frac{\pi}{2}$ it shows that, corresponds to a family of hyperbolas.

(*ii*) if $Z_0^2 - m^2\pi^2 > 0$ then $H^2 - AB < 0$ implies ellipse.

Therefore, Eqn. (29) represents a ellipse.

Hence,

$\frac{RZ_0}{\rho} - \frac{NX\,\mathrm{d}Z_0}{UR\rho} = m\pi$ or $(2m \pm 1)\frac{\pi}{2}$ it shows that, corresponds to a family of ellipses.

(*iii*) if $Z_0^2 - m^2\pi^2 = 0$ then $H^2 - AB = 0$ implies parabola.

Therefore, Eqn. (29) represents a parabola.

Hence,

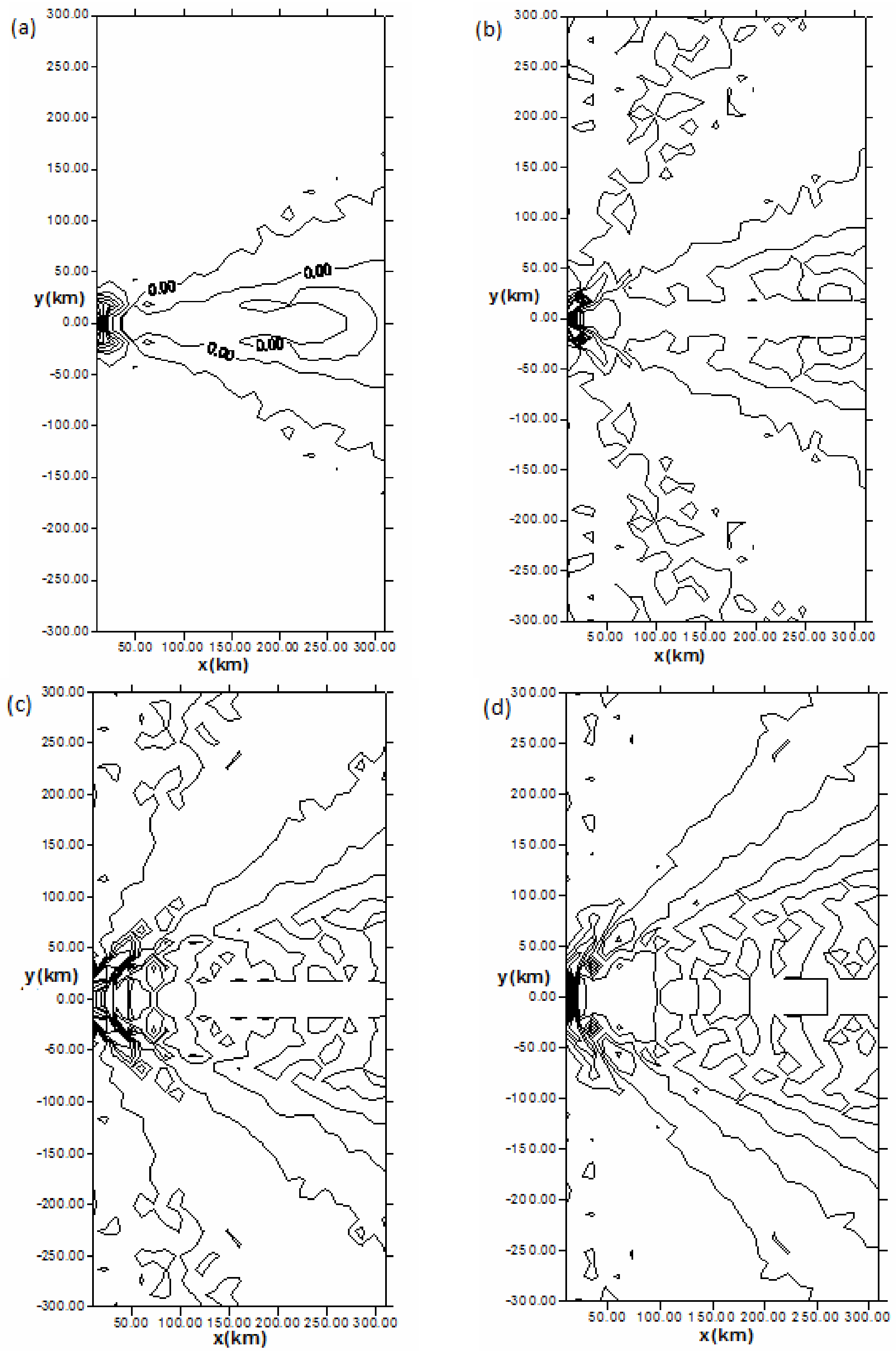
$\frac{RZ_0}{\rho} - \frac{NX\,\mathrm{d}Z_0}{UR\rho} = m\pi$ or $(2m \pm 1)\frac{\pi}{2}$ it shown that, corresponds to a family of parabolas.

In the study of Das (1964), nodal lines were concentric circles, which may be attributed to the geostrophic approximation made by him and the larger scale circular taken by him. In the study of Smith (1980) and Someiski (1981), were taking hydrostatic approximation, obtain parabolic shaped nodal lines for small Gaussian hill circular contour and the diameter of the circular obstacle is 50 km respectively. In the present study neither geostrophic nor hydrostatic approximation are made and furthermore, instead of a circular mountain we have taken a meso-scale elliptical barrier. Hence in this case we have two nodal lines, the first nodal lines are a system of hyperbola, which is in conformity with the earlier findings of Wurtele (1957) and also for the different values of m, we get the nodal lines as system of circle, system of hyperbola, system of ellipse and system parabola.

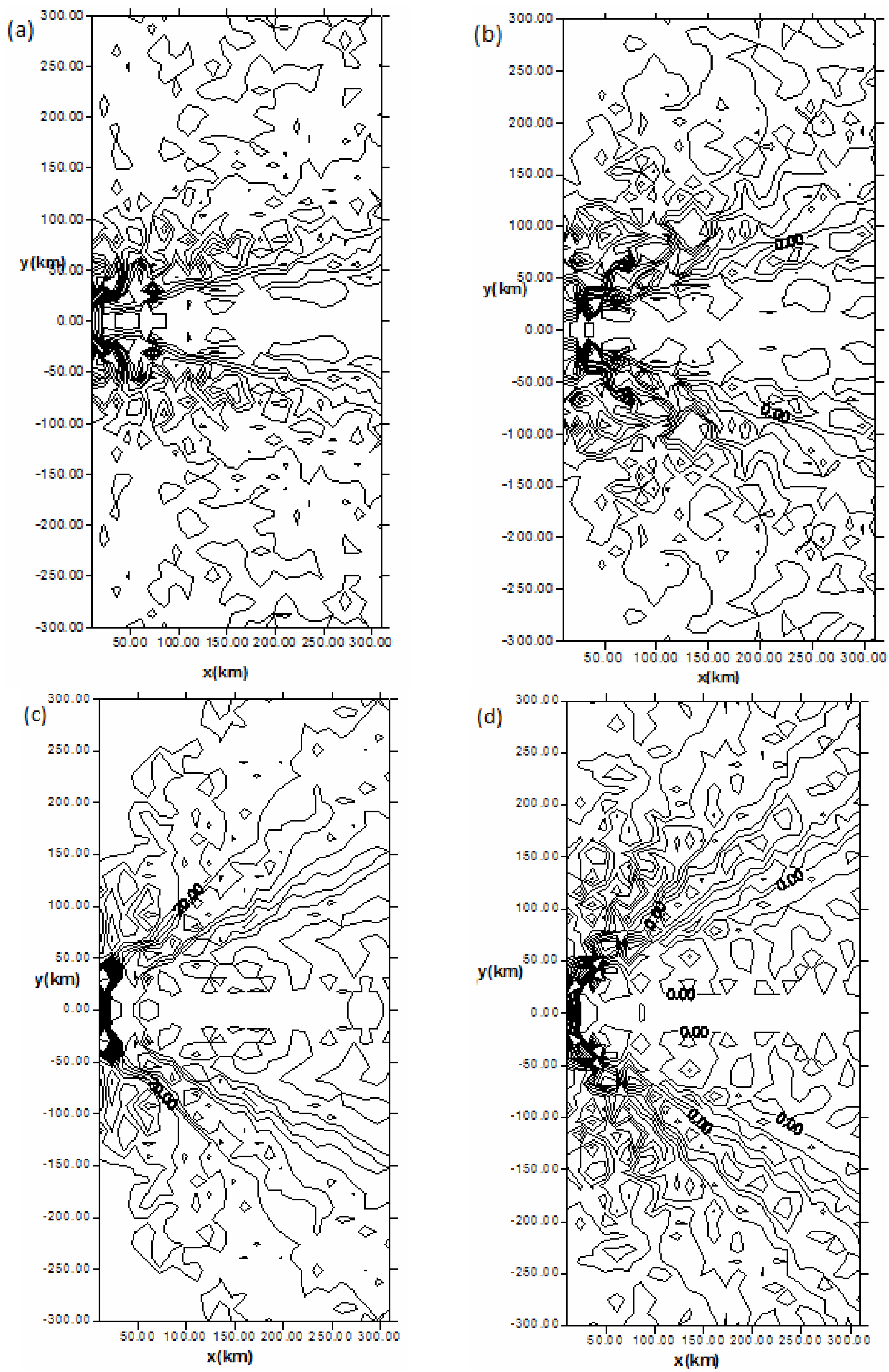
Now the above analytical result are verified for a typical Lee wave case across the Assam-Burma hills along east coast of India. De (1973) investigated that the air stream characteristic across Assam-Burma hills during winter was favorable for the occurrence of Lee waves. Using above equations both w' and η' are computed for this orographic barrier, mentioned above, for the Lee wave cases.

For the Assam-Burma hills we take a = 20 km, b = 50 km, d = 55 km, h_1 = 0.9 km, h_2 = 0.7 km.

Now for the typical case in the Assam-Burma hills the vertically averaged basic wind speed (U) is 10m/s and vertically averaged value of Brunt-Vaisalla frequency (N) for the basic flow is 0.01/s. It is seen that both w' and η' decay downstream of the barrier at all levels in the central plane. These results are also in conformity with the earlier findings of Wurtele (1957) and Lyra (1943). The train of oscillations in the downstream variation of w' and η' in the central plane may be attributed to the product of the damping factors *viz*., Bessel function of second kind of order zero and $\frac{x^2}{\left(x^2+z^2\right)^{3/2}}$ to the product of the two damping factors *viz*., Bessel function of second kind of order zero and $\frac{x^2}{\left(x^2+z^2\right)}$ respectively.



Figs. 3 (a-d). The contour of w′ at (a) 1.5 km, (b) 3 km, (c) 6 km and (d) 10 km level



Figs. 4 (a-d). The contour of n′ at (a) 1.5 km, (b) 3 km, (c) 6 km and (d) 10 km level

The contours of w' and η' computed using asymptotic method at 1.5 km, 3 km, 6 km, 10 km for ABH are shown in Figs. 3(a-d) and Figs. 4(a-d) respectively.

From these above contours it is inferred that at all levels the vertical velocity (w') and streamline displacement (η') tilt upstream (upwind) and spread laterally with height. Lateral spreading of the wave as shown in above Figs. 3(a-d) and Figs. 4(a-d) is due to presence of divergent part in the Lee wave. Gjevic and Marthinsen (1978) had also found that diverging type as well as transverse type Lee pattern analyzing satellite photograph to study the Lee wave patterns generated by isolated islands in the Norwegian sea and the Barents sea. In the Figs. 3(a-d) and Figs. 4(a-d) crescent shaped updraft regions. In the central plane of the all contours shown in fig. 3(a-b) and 4(a-d) are seen that a horizontal gap which is caused by the valley of some finite distance.

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Appendix 1

The Fourier Transform of the function $h(x,y)=\dfrac{h_1}{1+\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}}+\dfrac{h_2}{1+\dfrac{(x-d)^2}{a^2}+\dfrac{y^2}{b^2}}$ is given by

$$F\left[h(x,y)\right]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(x,y)e^{-i(kx+ly)}\mathrm{d}x\mathrm{d}y$$

$$\hat{h}(k,l)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{h_1}{1+\frac{x^2}{a^2}+\frac{y^2}{b^2}}+\frac{h_2}{1+\frac{(x-d)^2}{a^2}+\frac{y^2}{b^2}}\right]e^{-i(kx+ly)}\mathrm{d}x\mathrm{d}y$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{h_1}{1+\frac{x^2}{a^2}+\frac{y^2}{b^2}}e^{-i(kx+ly)}\mathrm{d}x\mathrm{d}y+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{h_2}{1+\frac{(x-d)^2}{a^2}+\frac{y^2}{b^2}}e^{-i(kx+ly)}\mathrm{d}x\mathrm{d}y$$

Putting $x = aX$, $y = bY$ for the first term and $x - d = aX$, $y = bY$ for the 2nd term

And $k=\dfrac{k'}{a}$, $l=\dfrac{l'}{b}$ for the both terms and we get the following

$$\hat{h}(k,l)=ab\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{h_1e^{-i(k'X+l'Y)}}{1+X^2+Y^2}dXdY+abe^{-i\frac{k'd}{a}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{h_2e^{-i(k'X+l'Y)}}{1+X^2+Y^2}dXdY$$

$$\hat{h}(k,l)=ab(h_1+h_2e^{-ikd})\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-i(k'X+l'Y)}}{1+X^2+Y^2}dXdY$$

Putting X = r cos θ, Y = r sin θ and k' = K cos α, l = K sin α we get

$$\hat{h}(k,l)=ab\left(h_1+h_2e^{-ikd}\right)\int_0^{\infty}\int_0^{2\pi}\frac{e^{-irK\cos(\theta-\alpha)}}{1+r^2}rdrd\theta$$

Now $\int_0^{2\pi}e^{-irK\cos(\theta-\alpha)}\mathrm{d}\theta=2\pi J_0(rK)$ [taken from Dutta *et al*., (2002)]

And $\int_0^{\infty}\dfrac{rJ_0(rK)}{1+r^2}\mathrm{d}r=K_0(K)$ [taken from Dutta *et al*., (2002)]

Where $J_0(x)$ and $K_0(x)$ are Bessel function of 1st and 2nd kind of order zero respectively.

Hence, $\widehat{h}\left(k,l\right) = 2\pi ab\left(h_1 + h_2 e^{-ikd}\right)K_0\left(K\right)$

Now $K^2 = k^2a^2 + l^2b^2$

Therefore,

$$\widehat{h}\left(k,l\right) = 2\pi ab\left(h_1 + h_2 e^{-ikd}\right)K_0\left(\sqrt{k^2a^2 + l^2b^2}\right)$$

Appendix 2

Now to evaluate the integral I_1 and I_2 we follow the method of approximations of double integrals of functions of large numbers as given by Hsu (1948). In the theorem of this paper, he has shown that if $\phi\left(x,y\right), h\left(x,y\right)$ and $f\left(x,y\right) = e^{h(x,y)}$ be continuous functions defined on a region S such that

(*i*) $\phi\left(x,y\right), \left[f\left(x,y\right)\right]^n$ is absolutely integrable over S for n = 0, 1, 2.............

(*ii*) $\dfrac{\partial f}{\partial x}, \dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial f}{\partial y}, \dfrac{\partial^2 f}{\partial y^2}$ exist and continuous over S.

(*iii*) $h(x, y)$ has an absolute maximum value at an interior pt (x_0, y_0) such that

$$\text{At } \left(x_0, y_0\right), \frac{\partial h}{\partial x} = \frac{\partial h}{\partial y} = 0, \frac{\partial^2 h}{\partial x^2}\frac{\partial^2 h}{\partial y^2} - \left(\frac{\partial^2 h}{\partial x \partial y}\right)^2 > 0$$

(*iv*) $\phi\left(x,y\right)$ is continuous at (x_0, y_0) and $\phi\left(x_0, y_0\right) \neq 0$. Now if C be an analytic curve passing through the point (x_0, y_0), such that the region S is divided into two sub regions S_1 and S_2. Then the integral $\iint \phi\left(x,y\right)\left[f\left(x,y\right)\right]^n ds$ taken over either of S_1 and S_2 is asymptotic to

$$\frac{\pi\phi\left(x_0, y_0\right)\left[f\left(x_0, y_0\right)\right]^n}{\sqrt[n]{\left[\dfrac{\partial^2 h}{\partial x^2}\dfrac{\partial^2 h}{\partial y^2} - \left(\dfrac{\partial^2 h}{\partial x \partial y}\right)^2\right]_{\left(x_0, y_0\right)}}}$$

Now to evaluate I_1 the following transformations

$X = r\cos\theta, Y = r\sin\theta$ and $\lambda = \tau\sin\psi$ are made.

Here τ runs from 0 to ∞, and θ runs from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$. So that I_1 transformed

$$I_1 = \int_0^\infty \int_{-\pi/2}^{\pi/2} \left(h_1 + h_2 e^{-i\frac{dN\tau\cos\psi}{U}}\right)K_0\left(\frac{N}{U}\tau\sqrt{a^2\cos^2\psi + b^2\sin^2\psi}\right)e^{i\left[\tau r\cos\left(\psi-\theta\right) + Z\sqrt{\sec^2\psi - \tau^2}\right]}\tau d\tau d\psi$$

Now evaluate the above integral by Hsu's theorem we get

$$h(\tau,\psi) = i\left[\tau r\cos(\psi-\theta) + z\sqrt{\sec^2\psi - \tau^2}\right]$$

$$\phi(\tau,\psi) = \tau\left(h_1 + h_2 e^{-i\frac{dN\tau\cos\psi}{U}}\right)K_0\left(\frac{N}{U}\tau\sqrt{a^2\cos^2\psi + b^2\sin^2\psi}\right)$$

$f(\tau,\psi) = e^{h(\tau,\psi)}$ here n = 1

Clearly ϕ, h, f satisfy all the conditions stated in Hsu's theorem.

Also $\frac{\partial h}{\partial \tau} = 0$ and $\frac{\partial h}{\partial \psi} = 0$ gives

$$\tau = \frac{xz\sqrt{\rho^4 + X^2Y^2}}{\rho^3 R} = \tau_0 \text{ (say)}$$

$$\psi = \tan^{-1}\left(-\frac{XY}{\rho^2}\right) = \psi_0 \text{ (say)}$$

Where $\rho^2 = Y^2 + Z^2, R^2 = X^2 + Y^2 + Z^2$

Now

$$\frac{\partial^2 h}{\partial \tau^2}\frac{\partial^2 h}{\partial \psi^2} - \left(\frac{\partial^2 h}{\partial \tau \partial \psi}\right)^2 = R^2\left[1 + 4\left(\frac{XYZR}{\rho^4 + X^2Y^2}\right)\right] > 0$$

Now $\phi(\tau_0,\psi_0) = \frac{XZ\sqrt{\rho^4 + X^2Y^2}}{\rho^3 R}\left(h_1 + h_2 e^{-i\frac{dNXZ}{\rho UR}}\right)K_0\left(\frac{XZN}{RU\rho^3}\sqrt{a^2\rho^4 + b^2X^2Y^2}\right)$

By Hsu's theorem we get from (20)

$$I_1 = \frac{\pi\phi(\tau_0,\psi_0)\left[f(\tau_0,\psi_0)\right]^n}{\sqrt[n]{\left[\frac{\partial^2 h}{\partial \tau^2}\frac{\partial^2 h}{\partial \psi^2} - \left(\frac{\partial^2 h}{\partial \tau \partial \psi}\right)^2\right](\tau_0,\psi_0)}} \quad \text{here } n = 1$$

i.e.,

$$I_1 = \frac{\pi\frac{XZ\sqrt{\rho^4 + X^2Y^2}}{\rho^3 R}\left(h_1 + h_2 e^{-i\frac{dNXZ}{\rho UR}}\right)K_0\left(\frac{XZN}{RU\rho^3}\sqrt{a^2\rho^4 + b^2X^2Y^2}\right)}{\sqrt{R^2\left[1 + 4\left(\frac{XYZR}{\rho^4 + X^2Y^2}\right)^2\right]}}e^{i\frac{ZR}{\rho}}$$

From (18) we get

$$\eta'_1\left(X,Y,Z\right)=\frac{abN^2}{2\pi U^2}\mathrm{Re}\left[\frac{\pi\dfrac{XZ\sqrt{\rho^4+X^2Y^2}}{\rho^3 R}\left(h_1+h_2 e^{-i\frac{dNXZ}{\rho UR}}\right)K_0\left(\dfrac{XZN}{RU\rho^3}\sqrt{a^2\rho^4+b^2X^2Y^2}\right)}{\sqrt{R^2\left[1+4\left(\dfrac{XYZR}{\rho^4+X^2Y^2}\right)^2\right]}}\right]e^{i\frac{ZR}{\rho}}$$

i.e.,

$$\eta'_1\left(X,Y,Z\right)=\frac{abN^2XZ}{2R^2U^2\rho^3}\left[\frac{\left(\rho^4+X^2Y^2\right)^{1/2}K_0\left(\dfrac{XZN}{RU\rho^3}\sqrt{a^2\rho^4+b^2X^2Y^2}\right)}{\sqrt{1+4\left(\dfrac{XYZR}{\rho^4+X^2Y^2}\right)^2}}\right]\left[h_1\cos\left(\frac{ZR}{\rho}\right)+h_2\cos\left(\frac{ZR}{\rho}-\frac{NXZd}{UR\rho}\right)\right]$$

Similarly it can be shown that from (21)

$$I_2=\frac{i\pi X^2Z^2\sqrt{\rho^4+X^2Y^2}\left(h_1+h_2 e^{-i\frac{dNXZ}{\rho UR}}\right)K_0\left(\dfrac{XZN}{RU\rho^3}\sqrt{a^2\rho^4+b^2X^2Y^2}\right)}{\sqrt[\rho^4R^3]{1+4\left(\dfrac{XYZR}{\rho^4+X^2Y^2}\right)^2}}$$

Now from (19) we get

$$w'_1\left(X,Y,Z\right)=\frac{abUN^3}{2\pi U^3}\mathrm{Re}\left[\frac{i\pi X^2Z^2\sqrt{\rho^4+X^2Y^2}\left(h_1+h_2 e^{-i\frac{dNXZ}{\rho UR}}\right)K_0\left(\dfrac{XZN}{RU\rho^3}\sqrt{a^2\rho^4+b^2X^2Y^2}\right)}{\sqrt[\rho^4R^3]{1+4\left(\dfrac{XYZR}{\rho^4+X^2Y^2}\right)^2}}\right]e^{i\frac{ZR}{\rho}}$$

i.e.,

$$w'_1\left(X,Y,Z\right)=-\frac{abN^3X^2Z^2}{2U^2\rho^4R^3}\left[\frac{\left(\rho^4+X^2Y^2\right)^{1/2}K_0\left(\dfrac{XZN}{RU\rho^3}\sqrt{a^2\rho^4+b^2X^2Y^2}\right)}{\sqrt{1+4\left(\dfrac{XYZR}{\rho^4+X^2Y^2}\right)^2}}\right]\left[h_1\sin\left(\frac{ZR}{\rho}\right)+h_2\sin\left(\frac{ZR}{\rho}-\frac{NXZd}{UR\rho}\right)\right]$$